Fast Algorithms for Nonlinear Multistage Stochastic Programs

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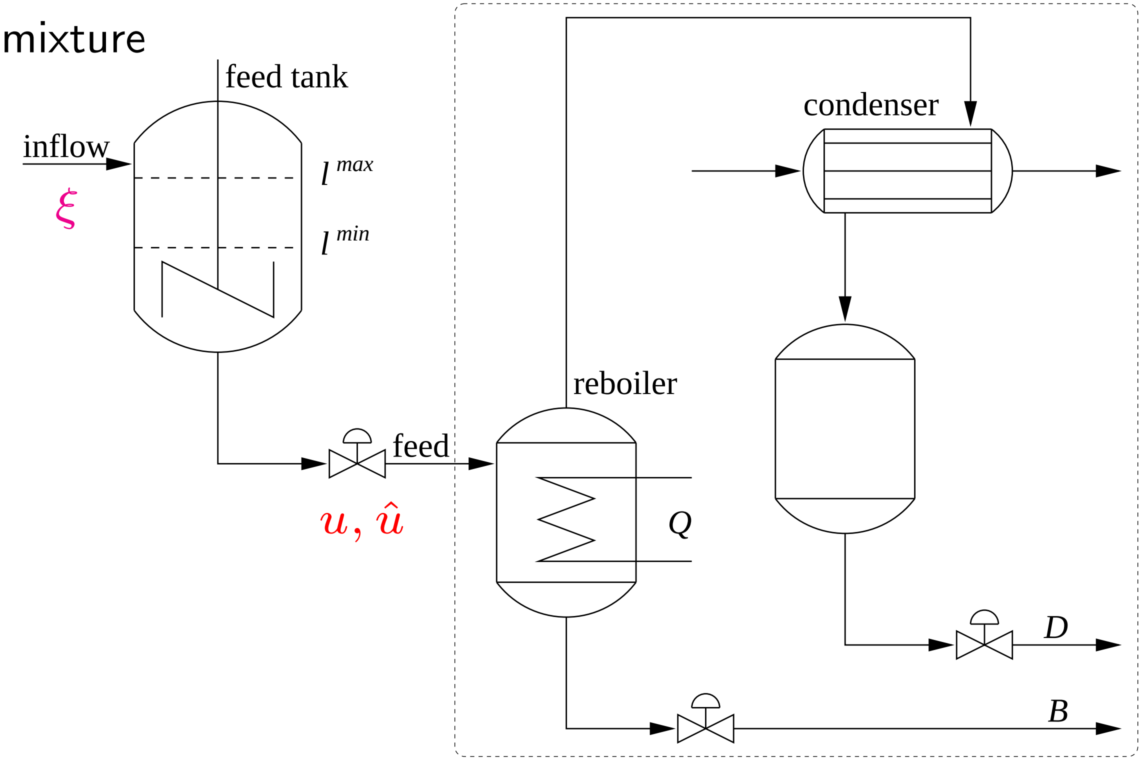
2005-10-20

Outline

- Introduction
- Algorithms
- Application
- Conclusions

Example 1: Distillation Process

Separation of methanol/water mixture with stochastic inflow



Goal

Tracking a desired (optimal) extraction profile without exceeding tank level bounds

Basic Assumption

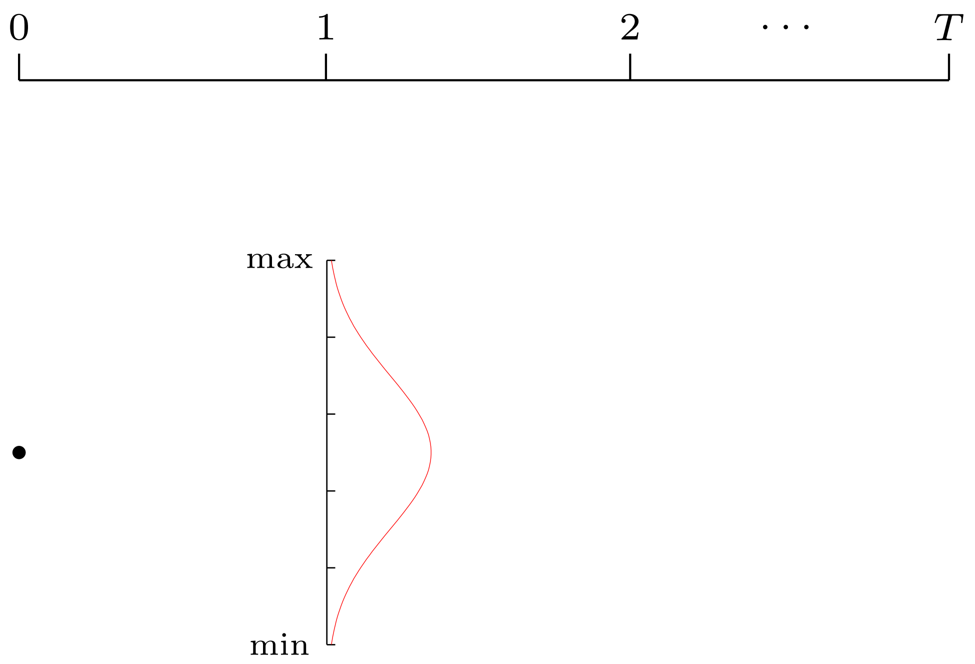
Probabilistic model of inflow process at hand

Cooperation: G. Wozny (TU Berlin), R. Henrion (WIAS Berlin) – Support: DFG

Refs: Eich-Soellner et al. 1997; Garcia et al. 1998; Li et al. 1998; Garrido, St. 2001; Henrion et al. 2001

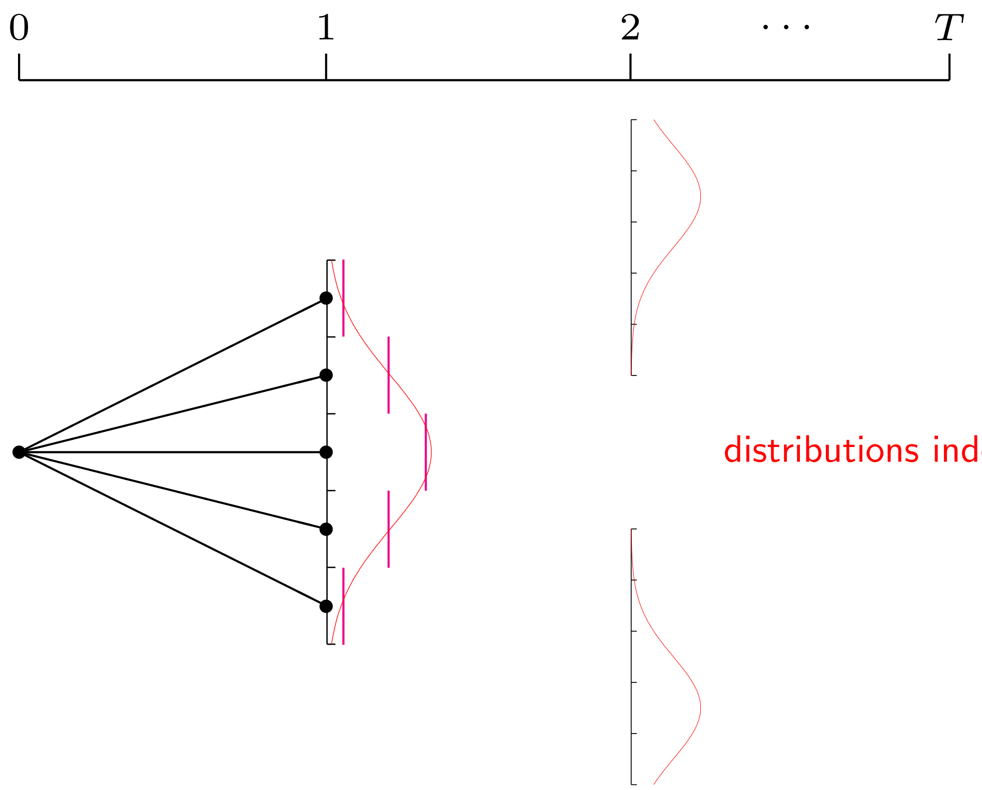
Example 1: Distillation Process

Scenario Tree Construction



Example 1: Distillation Process

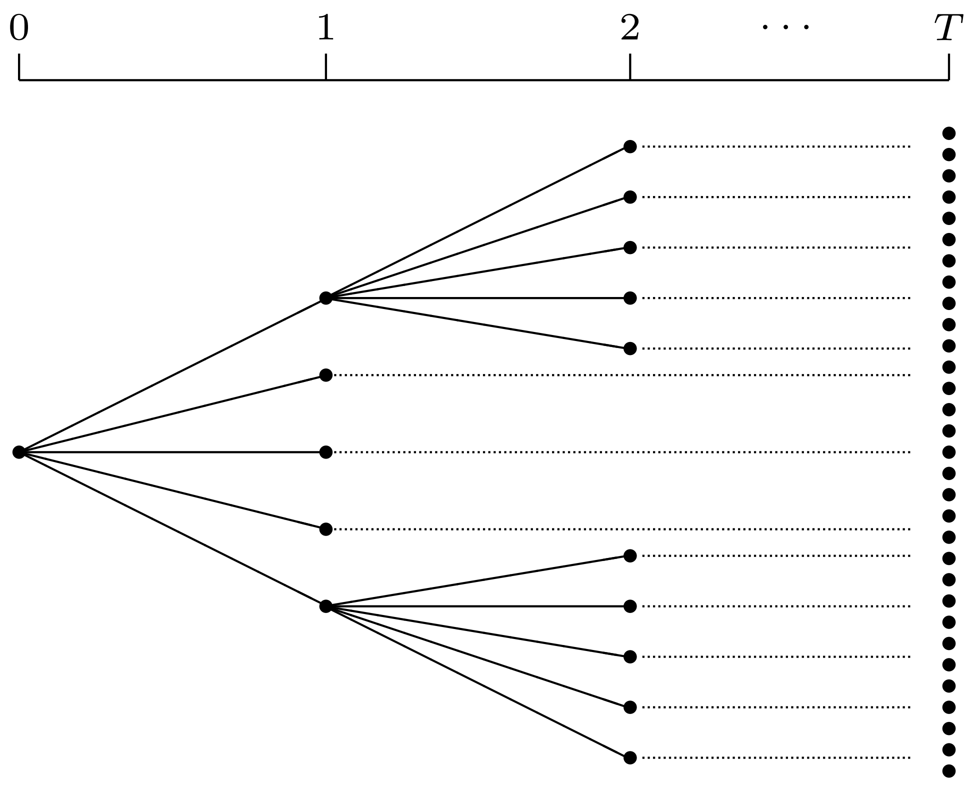
Scenario Tree Construction



distributions independent of decisions!

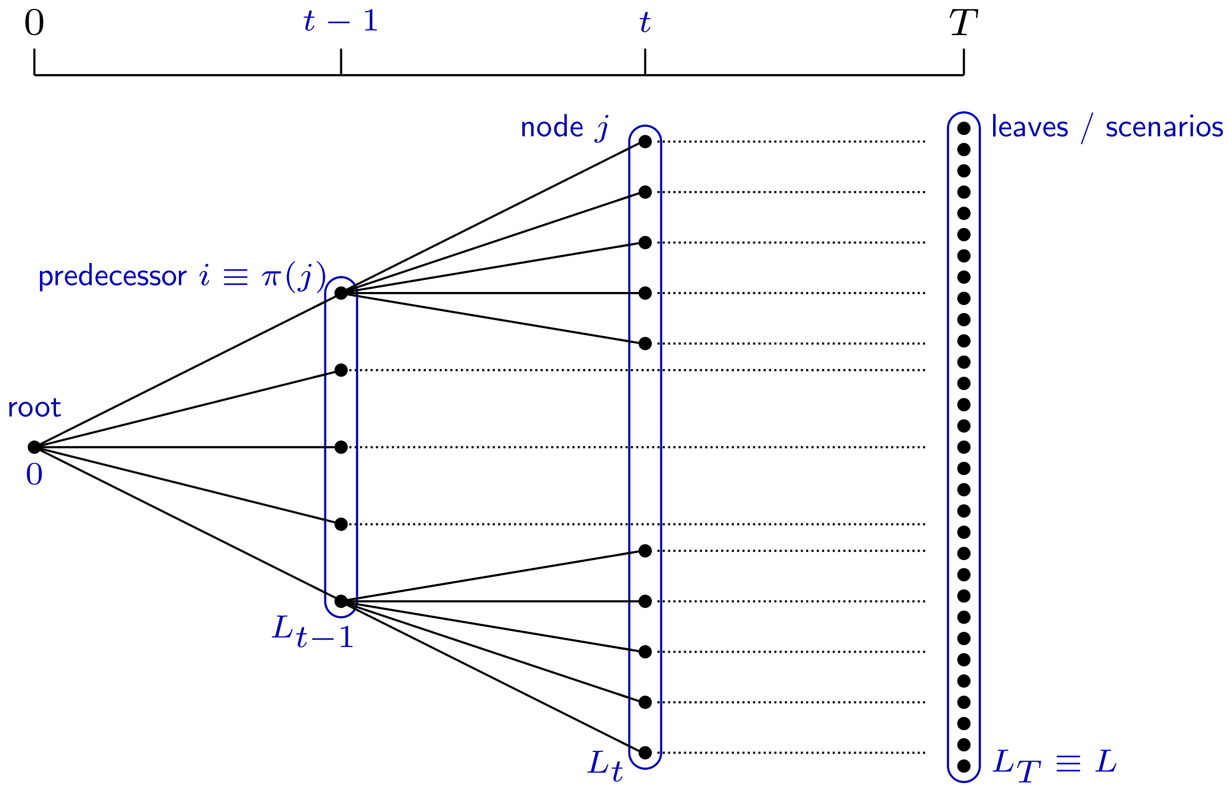
Example 1: Distillation Process

Scenario Tree $\mathcal{T} = (V, E)$



Example 1: Distillation Process

Scenario Tree $\mathcal{T} = (V, E)$: Notation



Node probability p_j $\quad\quad \sum_{j \in L_t} p_j = 1$ for $t = 0, \ldots, T$

Example 1: Distillation Process

Discrete Optimization Problem

x, u: Tank level, extraction

$\hat{u}$: Nominal extraction

$$\min_{(x,u)} \quad \sum_{t=0}^{T-1} \sum_{j \in L_t} p_j \left| u_j - \hat{u}_t \right|^2$$

subject to $\quad x_j = x_{\pi(j)} - u_{\pi(j)} + \xi_j \qquad j \in V$

$+$ terminal condition (stochastic)

$+$ simple bounds

Problem Size

$\dim(x, u)$	2
$\lvert V \rvert$	$10^5 \ldots 10^6$

Example 1: Integrated Process Dynamics

Differential variables

$$x = \begin{pmatrix} M_F \\ X_{1R} \\ X_{1C} \\ T_R \end{pmatrix}$$

M_F	molar holdup in feed tank
X_{1R}	liquid mole fraction of methanol at reboiler
X_{1C}	liquid mole fraction of methanol at total condenser
T_R	temperature at reboiler

Algebraic variables

$$z = \begin{pmatrix} V_R \\ Y_{1R} \end{pmatrix}$$

V_R	molar vapor flowrate
Y_{1R}	vapor mole fraction of methanol at reboiler

Control variables

$$u = \begin{pmatrix} F \\ Q \end{pmatrix}$$

F	molar feed flowrate
Q	heating power at reboiler

Relaxed DAE-IVP for inconsistent iterates (x_j, z_j) [Schulz, Bock, St. 1998]

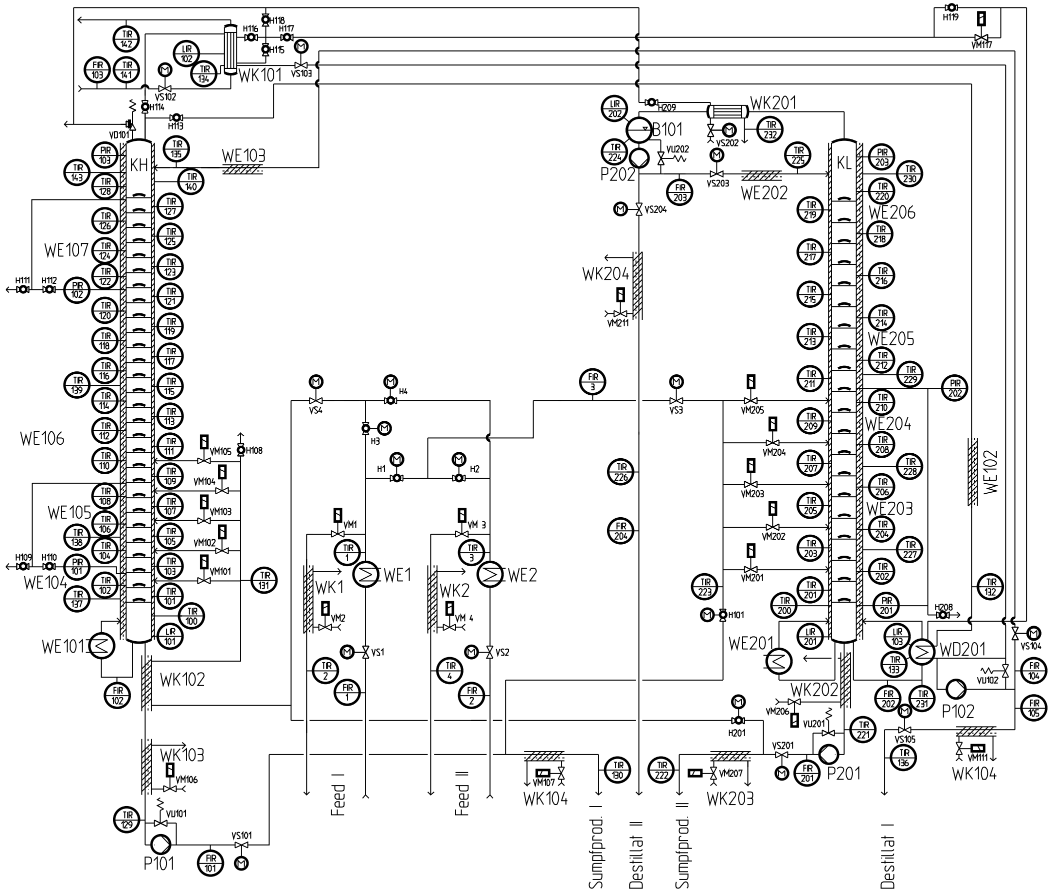
$$B(x)\dot{x} = f(x, z, u_i) + \xi_j e_1, \qquad x(\tau_i) = x_i,$$

$$0 = g(x, z) - g(x_i, z_i)e^{-\beta(\tau-\tau_i)}, \qquad z(\tau_i) = z_i.$$

Problem Size

$\dim(x, z, u)$	8
$\lvert V \rvert$	$10^5 \dots 10^6$

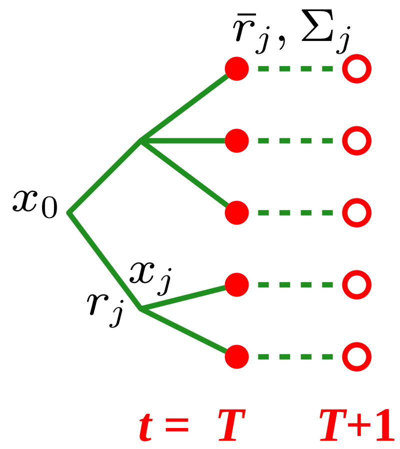
Example 1: Pilot System at TU Berlin



Example 2: Portfolio Management

DEVA (**D**ynamic **E**xpectation-**V**ariance **A**nalysis)

- Portfolio with n assets
- Invested amounts $x \in \mathbf{R}^n$, total wealth $e^* x$, $e = (1 \dots 1)$
- Random returns $r \in \mathbf{R}^n$, new wealth $r^* x = e^* \mathrm{Diag}(r) x$
- Investment horizon $[0, T+1]$
- Rebalancing at $t = 1 \dots T$
- DEVA:
  Minimize variance of total wealth ("risk")
  subject to prescribed expectation at $T+1$



Cooperation: K. Frauendorfer (ior/cf-HSG, U St. Gallen, Switzerland)

Refs: Frauendorfer 1995; St. 1998; Frauendorfer, Siede 2000; St. 2001

Example 2: Portfolio Management

DEVA Base Model

$\min\limits_x$	$\sum_{j \in L} p_j x_j^* (\Sigma_j + \bar{r}_j \bar{r}_j^*) x_j$	risk
subject to	$e^* x_0 = 1$	initial wealth
	$e^* x_j = r_j^* x_{\pi(j)} \qquad j \in V \setminus \{0\}$	rebalancing
	$\sum_{j \in L} p_j \bar{r}_j^* x_j = \rho$	target wealth

Problem Size

$\dim(x)$	10 . . . 20
$\lvert V \rvert$	$10^4 \ldots 10^5$

Algorithm	Storage	Runtime
IfU	$O(\lvert V \rvert^2)$	$O(\lvert V \rvert^3)$
ZIB	$O(\lvert V \rvert)$	$O(\lvert V \rvert)$

Problem Class: Overall Structure

Tree topology $\mathcal{T} = (V, E)$

Dynamic Optimization Problem

$$\begin{aligned} \min_{y} \quad & \sum_{j \in V} \mathsf{cost}_j(y_j) \\ \text{subject to} \quad & \mathsf{dyn}_j(y_{\pi(j)}, y_j) = 0 \qquad j \in V \\ & \sum_{j \in V} \mathsf{glob}_j(y_j) = 0 \\ & B_j y_j \geq b_j \qquad j \in V \end{aligned}$$

Special Case: Control Problems
$x_j = \mathsf{dyn}_j(x_{\pi(j)}, u_{\pi(j)})$ or $x_j = \mathsf{dyn}_j(x_{\pi(j)}, u_j)$

Structure: Nonlinear coupling: father-son

Problem Class: Overall Structure

Tree topology $\mathcal{T} = (V, E)$

Non-Markovian Dynamic Optimization Problem

$$\min_{y} \quad \sum_{j \in V} \mathrm{cost}_j(y_0, \ldots, y_{\pi(j)}, y_j)$$

$$\text{subject to} \quad \mathrm{dyn}_j(y_0, \ldots, y_{\pi(j)}, y_j) = 0 \qquad j \in V$$

$$\sum_{j \in V} \mathrm{glob}_j(y_0, \ldots, y_{\pi(j)}, y_j) = 0$$

$$B_j y_j \geq b_j \qquad j \in V$$

Special Case: Control Problems

$x_j = \mathrm{dyn}_j(x_0, u_0, \ldots, x_{\pi(j)}, u_{\pi(j)})$ or $x_j = \mathrm{dyn}_j(u_0, x_0, \ldots, u_{\pi(j)}, x_{\pi(j)}, u_j)$

Structure: Nonlinear coupling: all ancestors (path to root)

Overall Concept

Integrated Modeling and Solution Approach

- Tree-sparse NLP formulation: Exposing the rich structure
- Generic iterative algorithms: Preserving the structure
- Highly specialized linear algebra: Exploiting the structure

Generic Iterations

- SQP: Sequentiel Quadratic Programming
- IPM: Interior Point Method (primal-dual)
- Subproblem: Hierarchically structured KKT system

KKT Solver

Algebraic Structure: Tree-Sparse

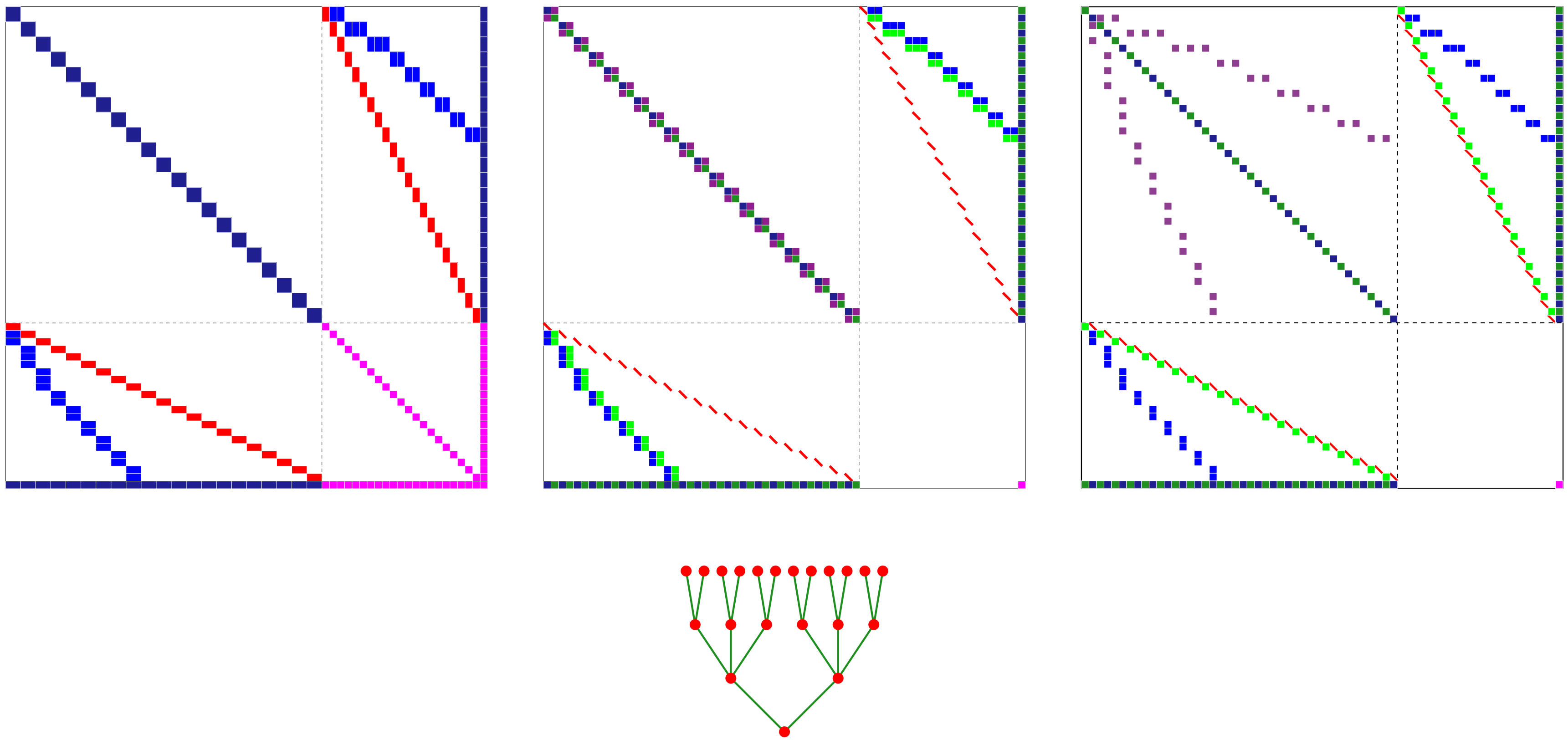
- NLP formulation: Hierarchical classification of restrictions
- Generic iterations . . .
- Linear algebra: Hierarchical structure of KKT matrix
  1. Superstructure: primal-dual (optimization)
  2. Block structure – coarse: tree topology (discretization)
  3. Block structure – fine: hierarchy of restrictions (modeling)
  4. Sub-block structure: problem specific (implementation)

Why classification of restrictions?

- KKT factorization with linear complexity $O(|V|)$
- Most general regularity assumptions
- Result: three problem types + associated factorizations

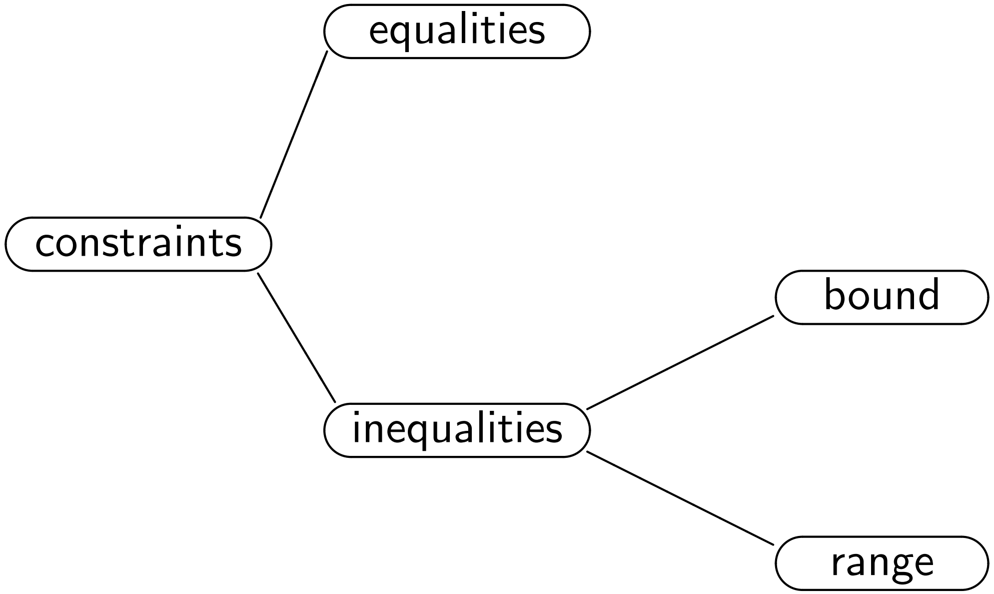
Overview KKT coarse structure

Implicit	Outgoing	Incoming
$\mathrm{dyn}(y_{\pi(j)}, y_j) = 0$	$x_j = \mathrm{dyn}(x_{\pi(j)}, u_{\pi(j)})$	$x_j = \mathrm{dyn}(x_{\pi(j)}, u_j)$

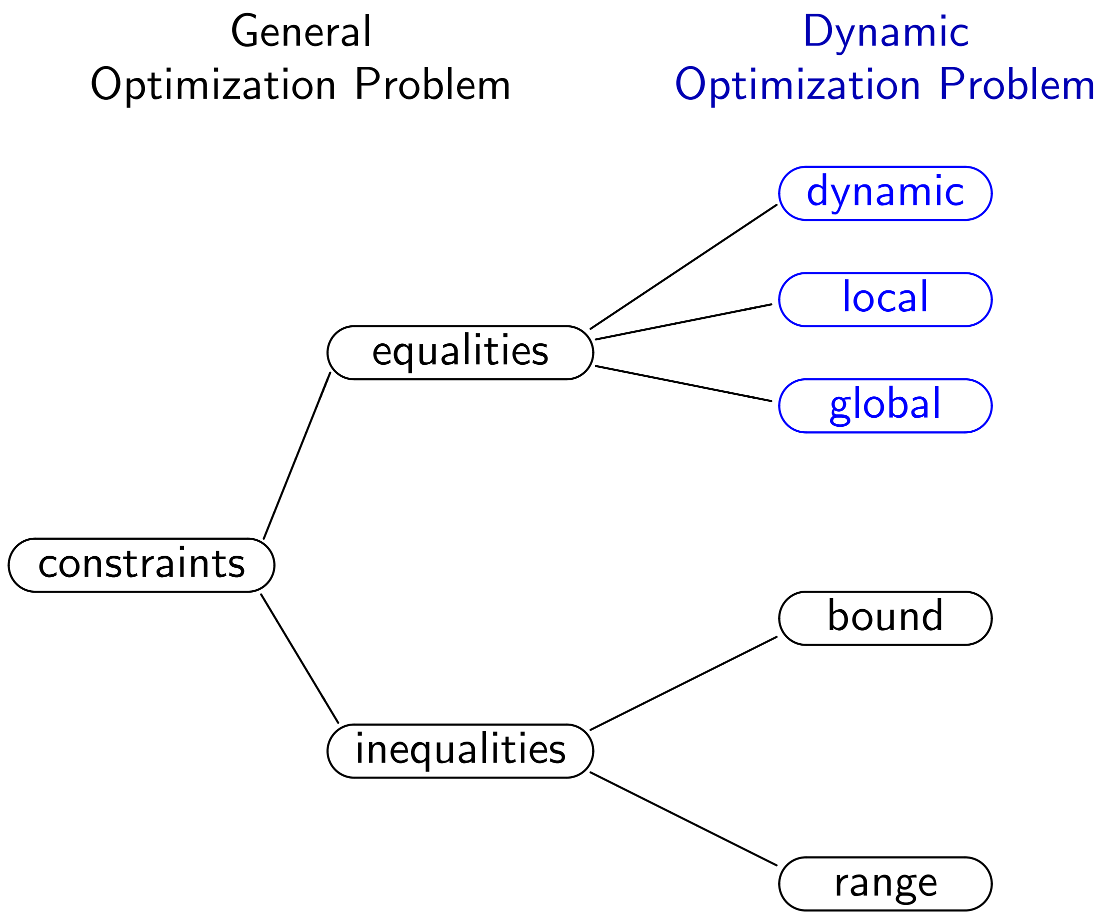


Classification of Restrictions

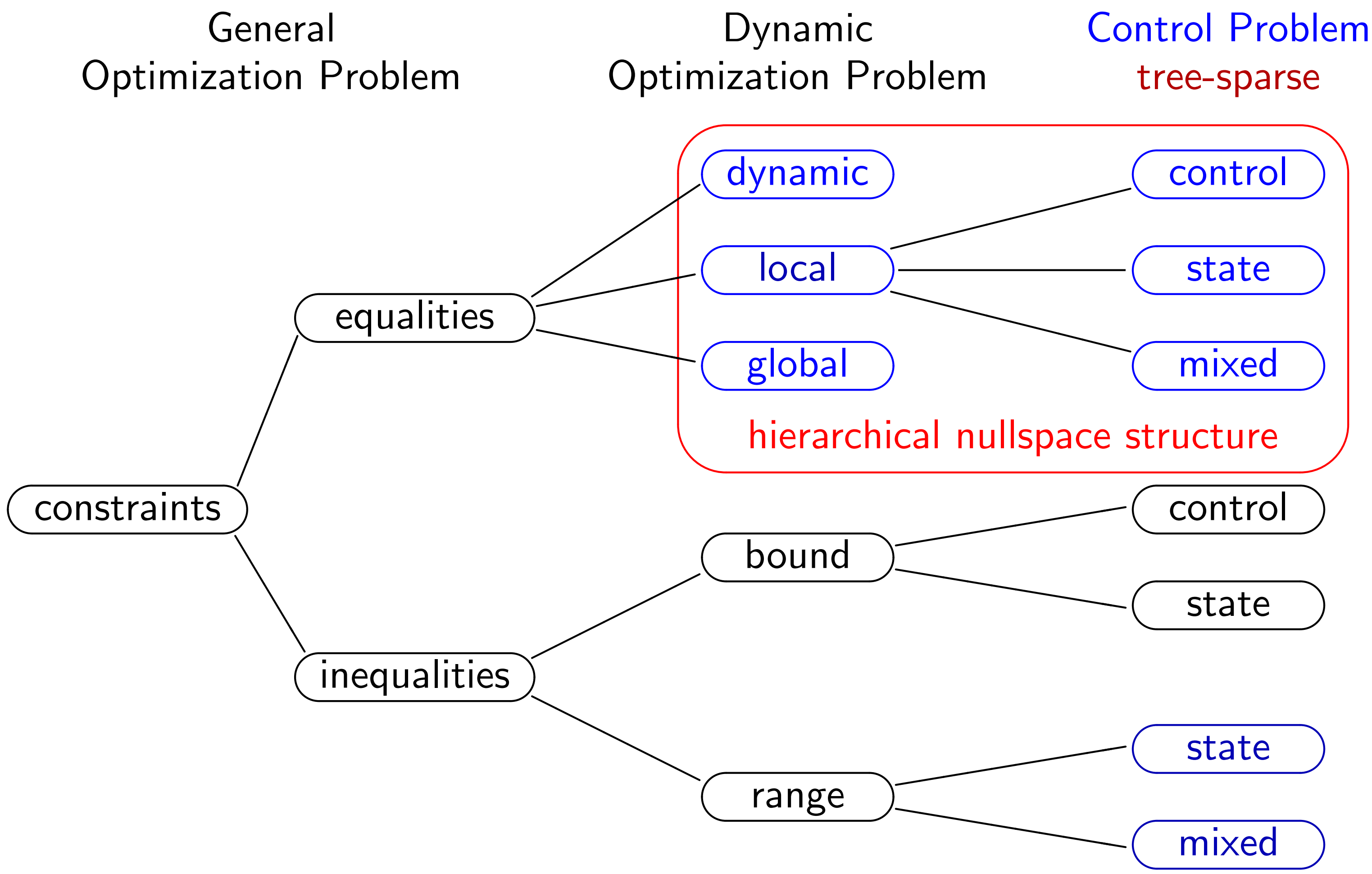
General
Optimization Problem



Classification of Restrictions



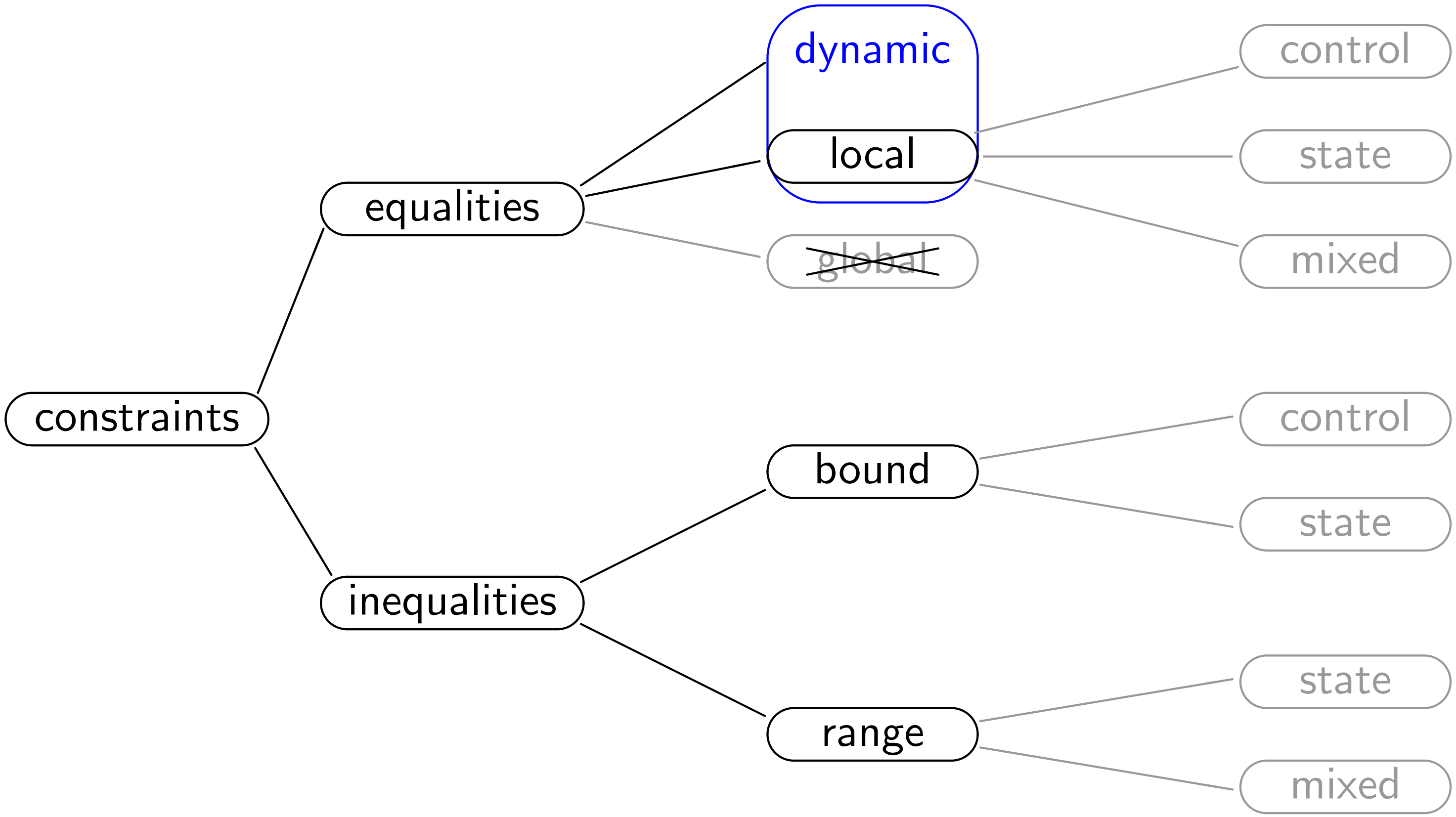
Classification of Restrictions



Earlier Solution Algorithms

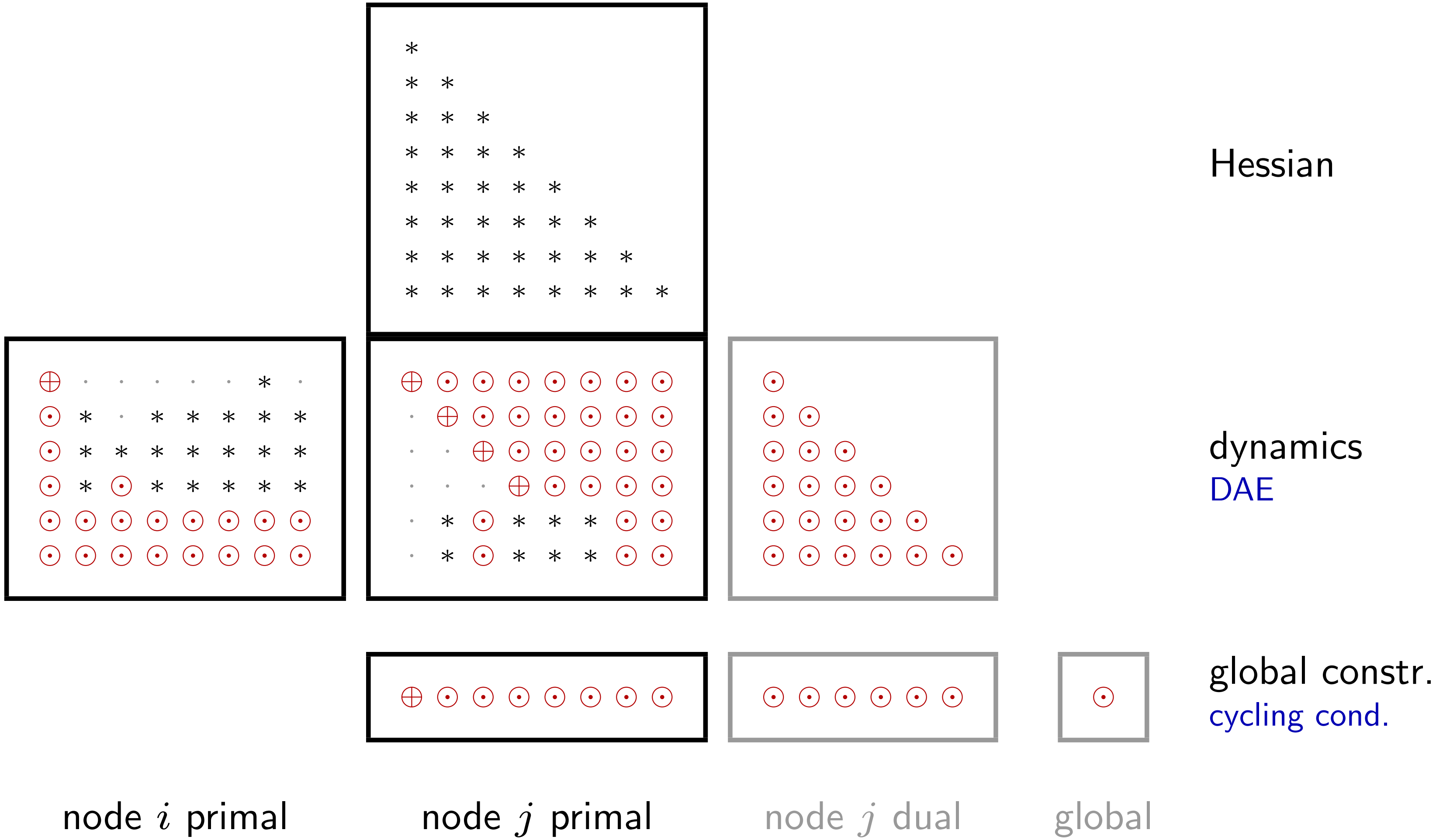
Stochastic Optimization

Birge & Qi 1988, . . . (two-stage), Schweitzer 1998



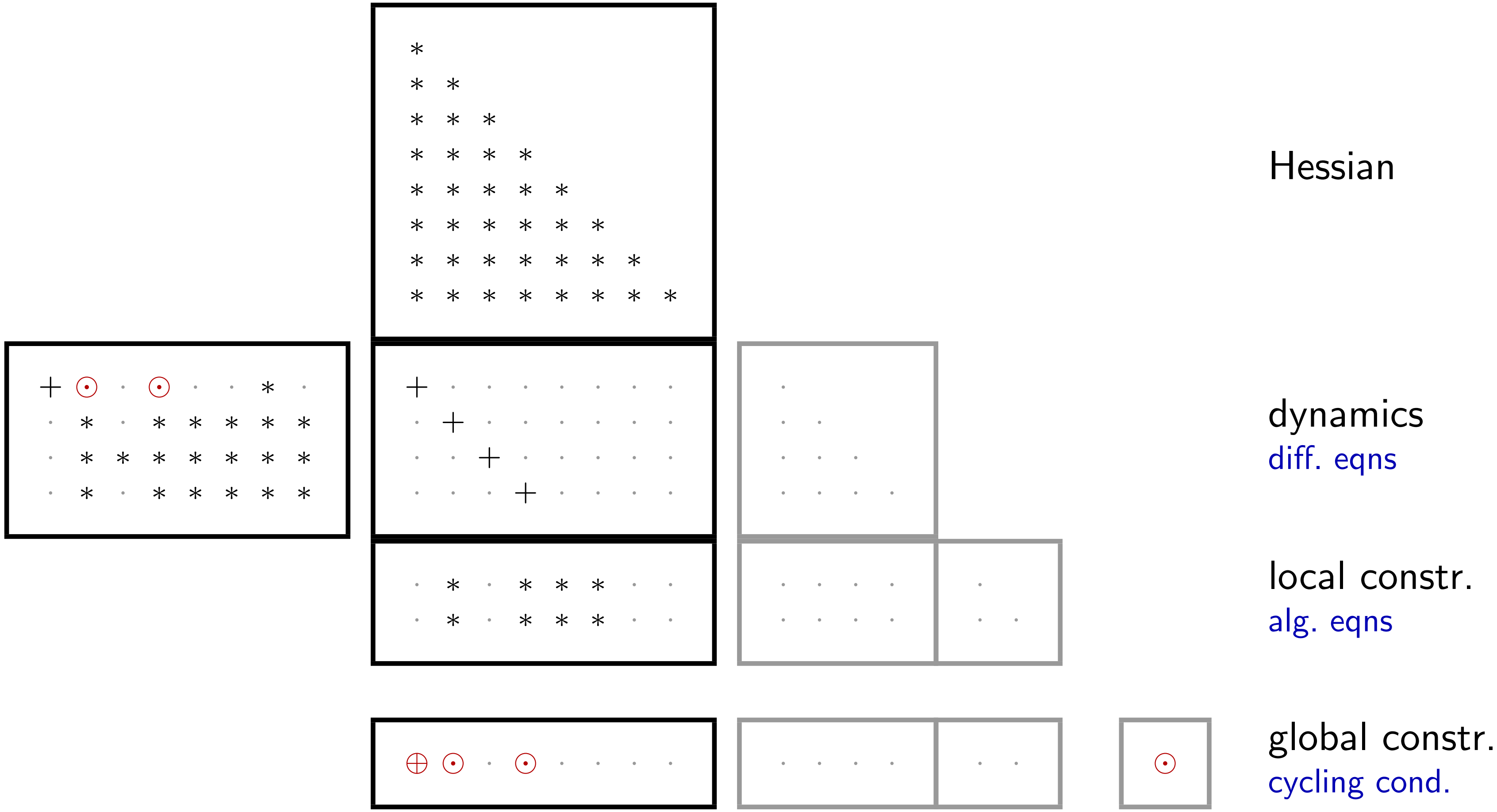
Algorithms

Fill-in for Example 1: standard form, Birge & Qi

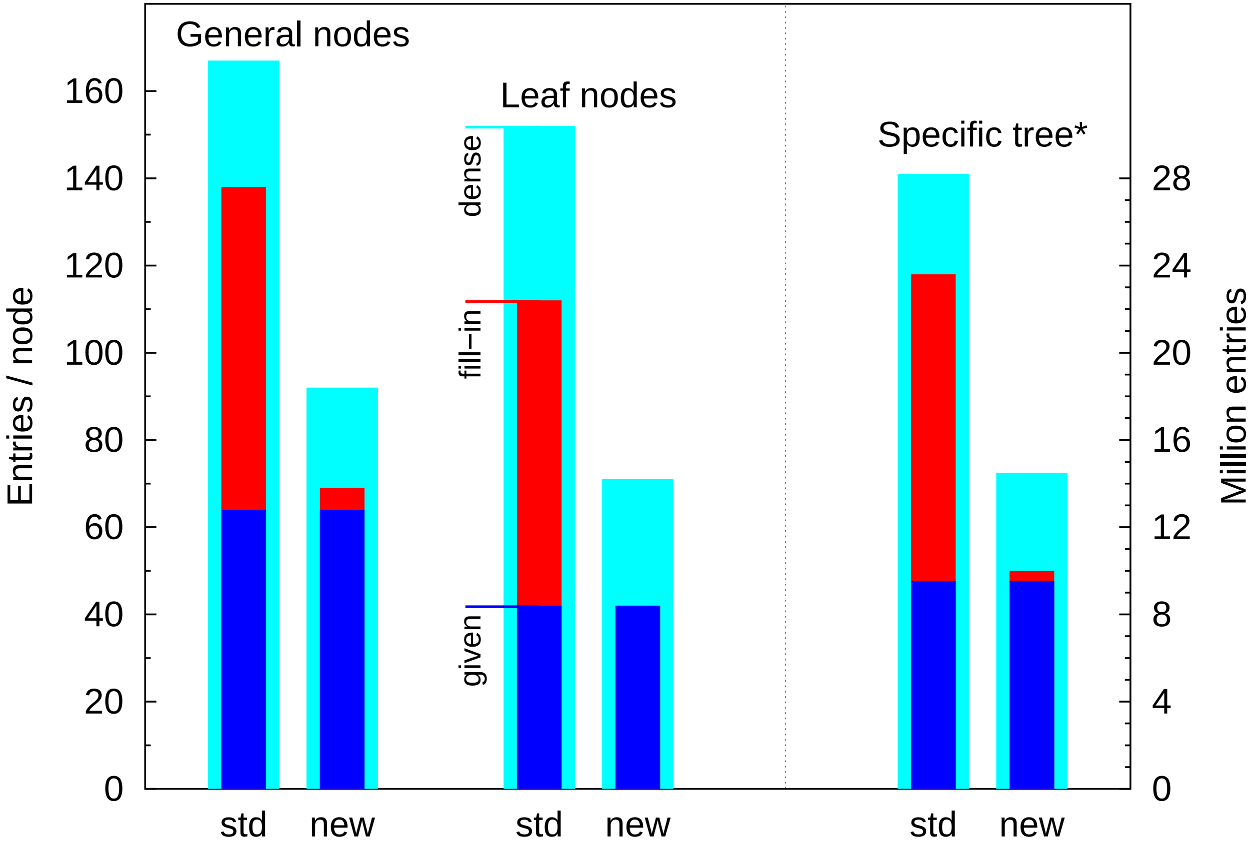


Algorithms

Fill-in for Example 1: tree-sparse control form – outgoing with local constraints



Fill-in for Example 1: comparison



* 8 periods, $5^7 \times 1 =$ 78,125 scenarios, 175,781 nodes
1,249,998 variables, 1,054,687 constraints

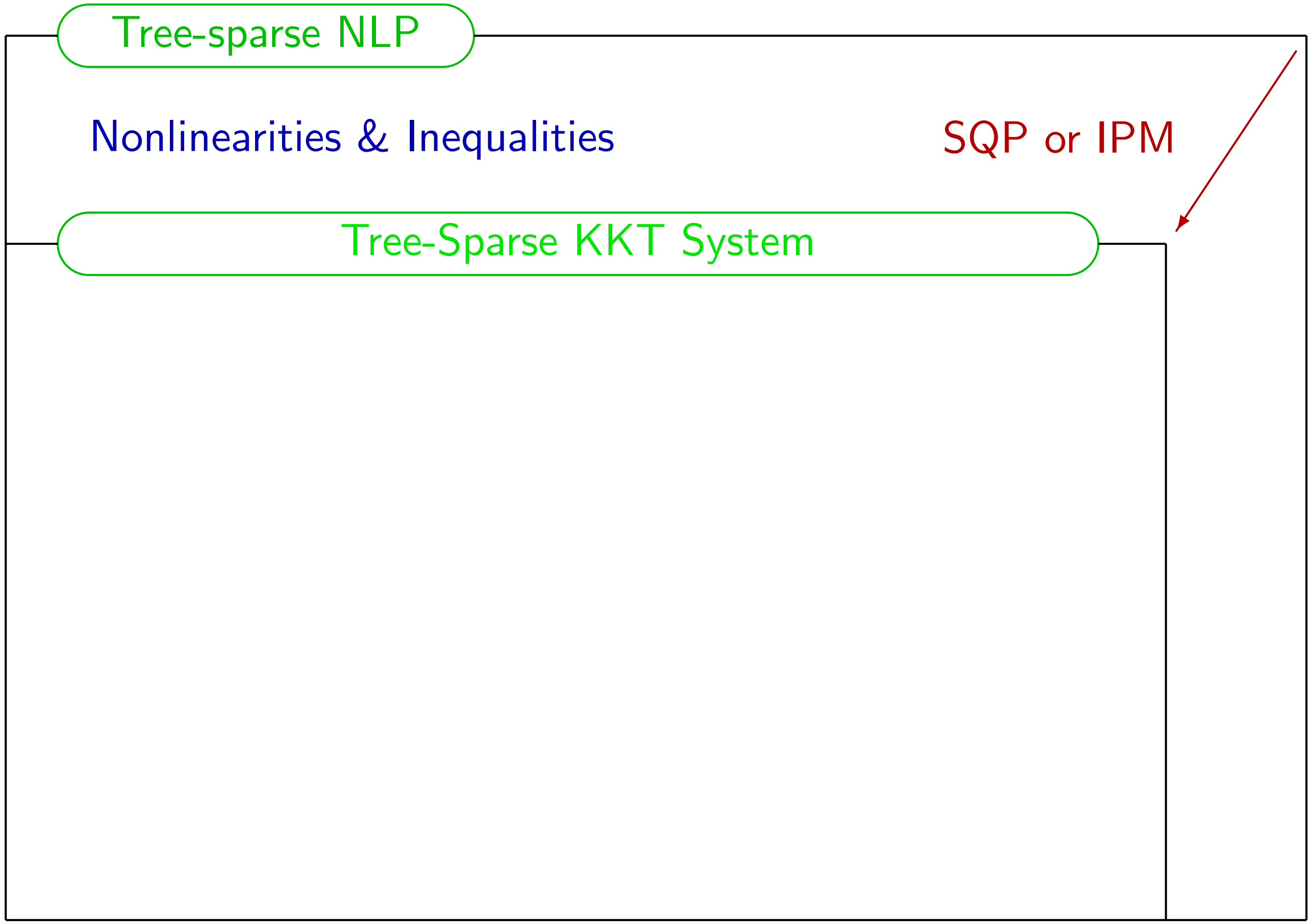
Hierarchic KKT Solver

Algorithmic Scheme

Elimination	Support	Method	Interpretation
local constraints	single nodes	local projections	primal
dynamics & objective	father & son*	tree recursion	primal-dual
global constraints	arbitrary node set	Schur complement Lagrange relaxation	dual

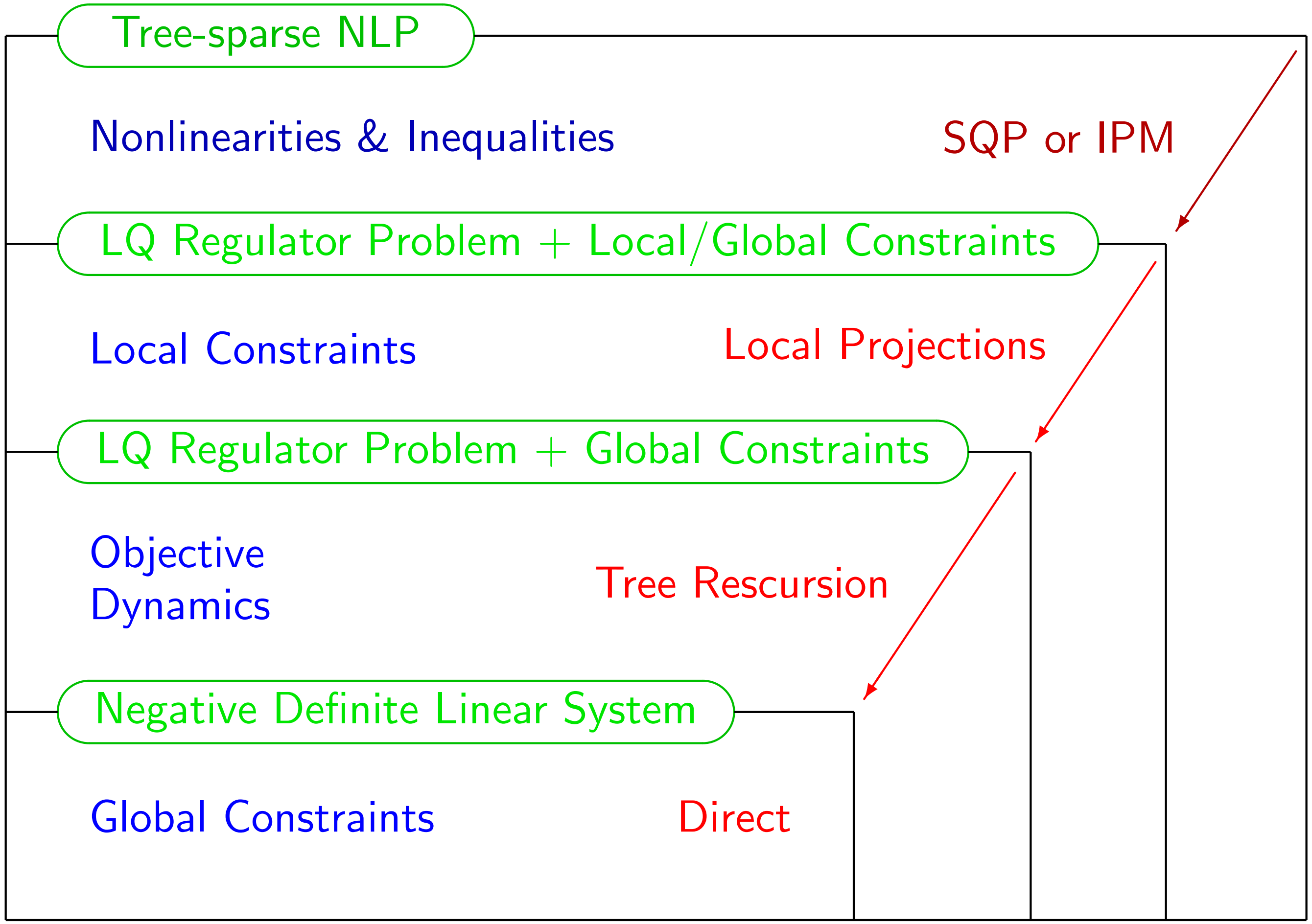
*possibly further ancestors

Control-Theoretic Interpretation



Algorithms

Control-Theoretic Interpretation



KKT Solver – Details

$$\mathcal{N} := N(A) \cap N(P_{\mathcal{B}_l \cup \mathcal{B}_u}) \cap N(P_{\mathcal{R}_l \cup \mathcal{R}_u} B).$$

(1) $\min_x \sum_{j \in V} p_j \varphi_j(x_j)$

(2) s.t. $G_j x_i + h_j = P_j x_j \quad \forall j \in V,$

(3) $F^x_j x_j + e^x_j = 0 \quad \forall j \in V,$

(4) $F^r_j x_j \in [r_{lj}, r_{uj}] \quad \forall j \in V,$

(5) $x \in [b_l, b_u],$

(6) $\sum_{j \in V} p_j F_j x_j + e_V = 0.$

(7) $$A = \begin{bmatrix} G \\ F^x \\ F \end{bmatrix}, \qquad B = \mathrm{Diag}(F^r_0, \dots, F^r_N) \equiv \begin{bmatrix} F^r_0 & & \\ & \ddots & \\ & & F^r_N \end{bmatrix},$$

(8) $$G = \begin{bmatrix} -P_0 & & & & & \\ G_1 & -P_1 & & & & \\ G_2 & & -P_2 & & & \\ & G_3 & & -P_3 & & \\ & G_4 & & & -P_4 & \\ & & G_5 & & & -P_5 \end{bmatrix}.$$

$$\mathcal{N}_j := N(P_{\mathcal{B}_{lj} \cup \mathcal{B}_{uj}}) \cap N(P_{\mathcal{R}_{lj} \cup \mathcal{R}_{uj}} F^r_j) \cap N(F^x_j) \cap \bigcap_{k \in S(j)} N(G_k).$$

(9) $\min_{x,s,t} \quad \varphi(x) - \beta \sum_\nu (\ln s^\nu_l + \ln s^\nu_u) - \beta \sum_\kappa (\ln t^\kappa_l + \ln t^\kappa_u)$

(10) s.t. $Ax + a = 0,$

(11) $Bx - t_l - r_l = 0,$

(12) $-Bx - t_u + r_u = 0,$

(13) $x - s_l - b_l = 0,$

(14) $-x - s_u + b_u = 0.$

$$L(x,s,t,z,u,v;\beta) = \varphi(x) - \beta \sum_\nu (\ln s^\nu_l + \ln s^\nu_u) - \beta \sum_\kappa (\ln t^\kappa_l + \ln t^\kappa_u) - z^*(Ax + a) - u_l^*(x - s_l - b_l) - u_u^*(-x - s_u + b_u) - v_l^*(Bx - t_l - r_l) - v_u^*(-Bx - t_u + r_u).$$

(15) $$\begin{bmatrix} \bar H & A^* \\ A & \end{bmatrix} \begin{bmatrix} \Delta x \\ -\Delta z \end{bmatrix} + \begin{bmatrix} \bar f \\ a \end{bmatrix} = 0.$$

$$\Phi := \Phi_l + \Phi_u \equiv S_l^{-1} U_l + S_u^{-1} U_u, \quad \Psi := \Psi_l + \Psi_u \equiv T_l^{-1} V_l + T_u^{-1} V_u,$$

$$\nabla \varphi(x) - A^* z - B^*(P^*_{\mathcal{R}_l} P_{\mathcal{R}_l} v_l - P^*_{\mathcal{R}_u} P_{\mathcal{R}_u} v_u) - (P^*_{\mathcal{B}_l} P_{\mathcal{B}_l} u_l - P^*_{\mathcal{B}_u} P_{\mathcal{B}_u} u_u) = 0,$$

$$\Phi_l := P^*_{\mathcal{B}_l} \bar\Phi_l P_{\mathcal{B}_l} \geq 0 \quad \text{where} \quad \bar\Phi_l := (P_{\mathcal{B}_l} S_l P^*_{\mathcal{B}_l})^{-1} (P_{\mathcal{B}_l} U_l P^*_{\mathcal{B}_l}) > 0.$$

$$\bar H = H + P^*_{\mathcal{B}_l} \bar\Phi_l P_{\mathcal{B}_l} + P^*_{\mathcal{B}_u} \bar\Phi_u P_{\mathcal{B}_u} + B^*(P^*_{\mathcal{R}_l} \bar\Psi_l P_{\mathcal{R}_l} + P^*_{\mathcal{R}_u} \bar\Psi_u P_{\mathcal{R}_u}) B.$$

$$N(\bar H) = N(H) \cap N(P_{\mathcal{B}_l}) \cap N(P_{\mathcal{B}_u}) \cap N(P_{\mathcal{R}_l} B) \cap N(P_{\mathcal{R}_u} B) = N(H) \cap N(P_{\mathcal{B}_l \cup \mathcal{B}_u}) \cap N(P_{\mathcal{R}_l \cup \mathcal{R}_u} B).$$

$$\mathrm{Diag}(\nabla^2 \varphi(x_0), \beta P^*_{\mathcal{B}_l} (P_{\mathcal{B}_l} S_{l0} P^*_{\mathcal{B}_l})^{-2} P_{\mathcal{B}_l}, \dots, \beta P^*_{\mathcal{R}_u} (P_{\mathcal{R}_u} T_{u0} P^*_{\mathcal{R}_u})^{-2} P_{\mathcal{R}_u}).$$

1

2

$$\min_{\Delta x} \frac{1}{2} \Delta x^* \bar H \Delta x + \bar f^* \Delta x \quad \text{s.t.} \quad A \Delta x + \bar a = 0$$

(16) $\bar H_j := \nabla^2 \varphi_j(x_j) + \Phi_j + F^{r*}_j \Psi_j F^r_j \geq 0.$

(17) $\min_{\Delta x} \sum_{j \in V} \frac{1}{2} \Delta x_j^* p_j \bar H_j \Delta x_j + p_j \bar f_j^* \Delta x_j$

(18) s.t. $G_j \Delta x_i + \bar h_j = P_j \Delta x_j,$

(19) $F^x_j \Delta x_j + \bar e^x_j = 0,$

(20) $\sum_{j \in V} p_j F_j \Delta x_j + \bar e_V = 0.$

$$L(x, \lambda, \mu^x, \mu) = \sum_{j \in V} p_j L_j(x_i, x_j, \lambda_j, \mu^x_j, \mu)$$

$$L_j(x_i, x_j, \lambda_j, \mu^x_j, \mu) = \frac{1}{2} x_j^* H_j x_j + f_j^* x_j - \lambda_j^*(G_j x_i + h_j - P_j x_j) - \mu^{x*}_j (F^x_j x_j + e^x_j) - \mu^*(F_j x_j + e_V).$$

(21) $H_j x_j + P^*_j \lambda_j - \sum_{k \in S(j)} G^*_k \lambda_k - F^{x*}_j \mu^x_j - F^*_j \mu = f_j \quad \forall j \in V,$

(22) $G_j x_i - P_j x_j = h_j \quad \forall j \in V,$

(23) $F^x_j x_j = e^x_j \quad \forall j \in V,$

(24) $\sum_{j \in V} F_j x_j = e_V.$

(25) $$\begin{bmatrix} H_{22} & G_2^* & F_2^* \\ G_2 & & \\ F_2 & & \end{bmatrix} \begin{bmatrix} x_2 \\ -\lambda \\ -\mu \end{bmatrix} = \begin{bmatrix} \bar f_2 \\ \bar h \\ \bar e_V \end{bmatrix}.$$

$$\begin{pmatrix} x_{j1} \\ x_{j2} \end{pmatrix} := U_j x_j, \quad \begin{pmatrix} f_{j1} \\ f_{j2} \end{pmatrix} := U_j^{-*} f_j, \quad \begin{pmatrix} H_{j11} & H^*_{j21} \\ H_{j21} & H_{j22} \end{pmatrix} := U_j^{-*} H_j U_j^{-1}.$$

(26) $-\mu^x_j = (L^x_j)^{-*} \Big[-H^*_{j21} x_{j2} - P^*_{j1} \lambda_j + \sum_{k \in S(j)} G^*_{k1} \lambda_k + F^*_{j1} \mu + (f_{j1} - H_{j11} x_{j1}) \Big].$

(27) $H_{j22} x_{j2} + P^*_{j2} \lambda_j - \sum_{k \in S(j)} G^*_{k2} \lambda_k - F^*_{j2} \mu = \bar f_{j2} \quad \forall j \in V,$

(28) $G_{j2} x_{i2} - P_{j2} x_{j2} = \bar h_j \quad \forall j \in V,$

(29) $\sum_{j \in V} F_{j2} x_{j2} = \bar e_V.$

$$\begin{bmatrix} -\hat Y & \hat Z^* \\ \hat Z & -\hat X_V \end{bmatrix} := \begin{bmatrix} -Y & Z^* \\ Z & -X_V \end{bmatrix} - \begin{bmatrix} G_2 \\ F_2 \end{bmatrix} H_{22}^{-1} [G_2^* \ \ F_2^*].$$

$$x_{j2} = H_{j22}^{-1}(-P^*_{j2} \lambda_j + F^*_{j2} \mu + \bar f_{j2}), \quad -\lambda_j = \hat Y_j^{-1}(G_{j2} x_{i2} - \hat Z_j^* \mu - \hat h_j).$$

$$N(\bar H_j) = N(H_j) \cap N(P_{\mathcal{B}_{lj} \cup \mathcal{B}_{uj}}) \cap N(P_{\mathcal{R}_{lj} \cup \mathcal{R}_{uj}} F^r_j).$$

(30) $H_{j22} \Big| \bigcap_{k \in S(j)} N(G_{k2}).$

$$\tilde H_j = H_{j22} + \sum_{k \in S(j)} G^*_{k2} \hat Y_k^{-1} G_{k2},$$

$$\begin{bmatrix} H_{22} & G_2^* \\ G_2 & \end{bmatrix} \begin{bmatrix} x_2 \\ -\lambda \end{bmatrix} = \begin{bmatrix} \bar f_2 + F_2^* \mu \\ \bar h \end{bmatrix}$$

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TABLE 1. Node operations of the tree-sparse SC method.

Step	Factorization	Transformation	Substitution
(1)	$F^x_j = L^x_j (I \ \ 0) U_j$	$x_{j1} = (L^x_j)^{-1} e^x_j$	$-\mu^x_j = (L^x_j)^{-1} \bar\mu^x_j$
(2)	$\begin{pmatrix} H_{j11} & H^*_{j21} \\ H_{j21} & H_{j22} \end{pmatrix} = U_j^{-*} H_j U_j^{-1}$	$\begin{pmatrix} f_{j1} \\ f_{j2} \end{pmatrix} = U_j^{-*} f_j$	$x_j = U_j^{-1} \begin{pmatrix} x_{j1} \\ x_{j2} \end{pmatrix}$
(3)		$\bar f_{j1} = f_{j1} - H_{j11} x_{j1}$	
(4)		$\bar f_{j2} = f_{j2} - H_{j21} x_{j1}$	$\bar\mu^x_j = -H^*_{j21} x_{j2} + \tilde\mu^x_j$
(5)*	$(G_{j1} \ \ G_{j2}) = G_j U_i^{-1}$	$\bar h_j = h_j - G_{j1} x_{i1}$	$\tilde f_{i1} = -G^*_{j1}(-\lambda_j) + \bar f_{i1}$
(6)	$(P_{j1} \ \ P_{j2}) = P_j U_j^{-1}$	$+ P_{j1} x_{j1}$	$\tilde\mu^x_j = +P^*_{j1}(-\lambda_j)$
(7)	$(F_{j1} \ \ F_{j2}) = F_j U_j^{-1}$	$\bar e_V = \bar e_{V'} - F_{j1} x_{j1}$	$-F^*_{j1}(-\mu) + \tilde f_{j1}$
(8)	$H_{j22} = L_j L_j^*$		
(9)	$\hat P_j = P_{j2} L_j^{-*}$		
(10)	$\hat F_j = F_{j2} L_j^{-*}$	$\hat f_j = L_j^{-1} \bar f_{j2}$	$x_{j2} = L_j^{-*}[$
(11)	$\hat Z_j = Z_j + \hat F_j \hat P_j^*$		
(12)	$\hat Y_j = Y_j + \hat P_j \hat P_j^*$	$\hat h_j = \bar h_j + \hat P_j \hat f_j$	$+\hat P_j^*(-\lambda_j)$
(13)	$\hat X_{V'} = X_V + \hat F_j \hat F_j^*$	$\hat e_{V'} = \bar e_V - \hat F_j \hat f_j$	$-\hat F_j^*(-\mu) + \hat f_j]$
(14)	$\hat Y_j = \hat L_j \hat L_j^*$		
(15)	$\tilde Z_j = \hat Z_j \hat L_j^{-*}$		
(16)	$\tilde G_j = \hat L_j^{-1} G_{j2}$	$\tilde h_j = \hat L_j^{-1} \hat h_j$	$-\lambda_j = \hat L_j^{-*}[$
(17)*	$\tilde F_i = F_{i2} + \tilde Z_j \tilde G_j$		
(18)*	$\tilde H_i = H_{i22} + \tilde G_j^* \tilde G_j$	$\tilde f_i = \bar f_{i2} + \tilde G_j^* \tilde h_j$	$\tilde G_j x_{i2} +$
(19)	$X_{V'} = \hat X_{V'} - \tilde Z_j \tilde Z_j^*$	$e_{V'} = \hat e_{V'} + \tilde Z_j \tilde h_j$	$\tilde Z_j(-\mu) - \tilde h_j]$
(20)	$X_\emptyset = L_\emptyset L_\emptyset^*$	$\hat e_\emptyset = L_\emptyset^{-1} e_\emptyset$	$-\mu = L_\emptyset^{-*}(-\hat e_\emptyset)$

$$X_\emptyset = [F_2 \ \ 0] \begin{bmatrix} U & V^* \\ V & W \end{bmatrix} \begin{bmatrix} F_2^* \\ 0 \end{bmatrix} = F_2 U F_2^*$$

$$\begin{bmatrix} U & V^* \\ V & W \end{bmatrix} := \begin{bmatrix} H_{22} & G_2^* \\ G_2 & \end{bmatrix}^{-1}.$$

$$\hat F_j \hat F_j^* - \hat F_j \hat P_j^* \hat Y_j^{-1} \hat P_j \hat F_j^* =: \hat F_j (I - Q) \hat F_j^*$$

entry	zero	+1	−1	generic
symbol	·	+	−	*

$$G_j = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & \cdot & * \\ * & \cdot & * & * & * & * & * & * & * & * \end{bmatrix}, \qquad F^x_j = \begin{bmatrix} - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & \cdot & \cdot \\ \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot & * & \cdot & \cdot \\ + & + & + & + & + & + & + & \ominus & - & \cdot \end{bmatrix},$$

$$P_j = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \oplus & \odot & \ominus \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \oplus & \oplus \end{bmatrix}, \qquad F^r_j = \begin{bmatrix} \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & * & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & * & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & * & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & * & \cdot & \cdot \end{bmatrix}.$$

$$P_{\mathcal{R}}(F^{r*}_j \Psi_j F^r_j) P^*_{\mathcal{R}} = \begin{bmatrix} * & \cdot & \cdot & \cdot & * \\ \cdot & * & \cdot & \cdot & * \\ \cdot & \cdot & * & \cdot & * \\ \cdot & \cdot & \cdot & * & * \\ * & * & * & * & * \end{bmatrix}.$$

$$U_j = \begin{bmatrix} I & W_j \\ & I \end{bmatrix}, \quad U_j^{-1} = \begin{bmatrix} I & -W_j \\ & I \end{bmatrix}, \quad W_j = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & * & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & * & \cdot & \cdot \\ + & + & + & + & \circledast & - & \cdot \end{bmatrix}.$$

Code Generator

Implementation

How to exploit block substructure?

- No implementation efficient on all problems
- Algorithm manually adaptable for specific problems
- Straightforward, but tedious & error-prone

$\longrightarrow$ Impractical: need software tool for custom code

Required Functionality

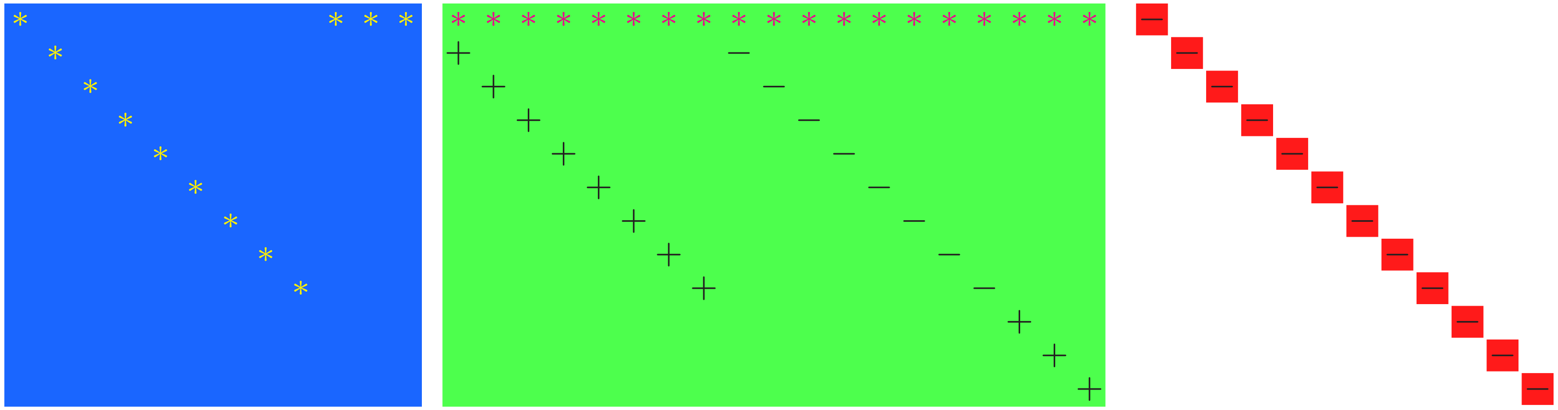
$$\left\{ \begin{array}{c} \text{access to} \\ \text{operations on} \end{array} \right\} \quad \left\{ \begin{array}{c} \text{vector} \\ \text{matrix} \\ \text{inverse} \end{array} \right\}$$

Realization

- A. Huțanu (Diploma thesis, 2002)
- No local projections yet

Entry Types

Example: entry types in DEVA



$\pm$	value ± 1 in every node and every problem instance
* (green)	identical value in every node
* (blue)	different value in every node

Test Examples

Finance Problems

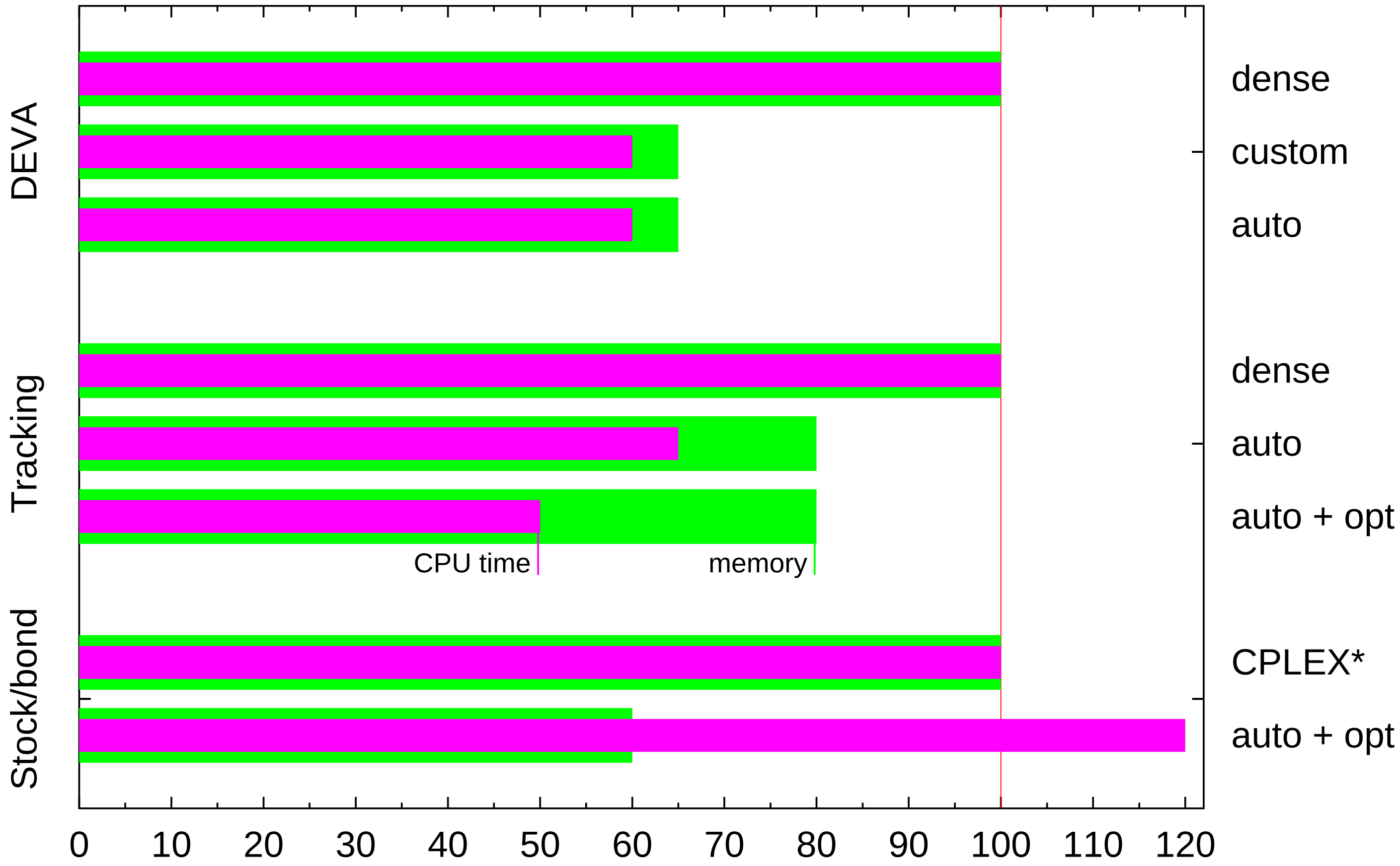
Problem	class	objective	constr.	dynamics	final cond.	granularity
Stock/bond	LP	linear	linear	implicit	—	very fine
Asset mgmt	CP	convex	linear	implicit	—	fine
DEVA	QP	quadratic	linear	incoming	global	medium

Chemical Engineering Problems

Problem	class	objective	constr.	dynamics	final cond.	granularity
Reactor	NLP	linear	ODE	outgoing	—	very fine
Tracking	QP	least squares	linear	outgoing	global/local	very fine
Distillation	NLP	linear	DAE	outgoing	global/local	fine

Code Generator

Savings



* No dense version available; CPLEX presolve eliminates all rows + columns

Portfolio Management

How it all began: single-period model (Markowitz 1952: *Portfolio Selection*)

$$\begin{aligned} \min_x \quad & x^* \Sigma x \\ \text{s.t.} \quad & e^* x = 1 \\ & \bar{r}^* x = \rho \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \min_x \quad & \sum_{j \in L} p_j x_j^* (\Sigma_j + \bar{r}_j \bar{r}_j^*) x_j \\ \text{s.t.} \quad & e^* x_0 = 1 \\ & e^* x_j = r_j^* x_{\pi(j)} \qquad j \in V \setminus \{0\} \\ & \sum_{j \in L} p_j \bar{r}_j^* x_j = \rho \end{aligned}$$

Portfolio Management

DEVA Application Model (+ riskless asset, cash flow, transaction costs, . . .)

$$\min_{(u,x)} \sum_{j\in L} p_j \begin{pmatrix} x_j^c \\ x_j \end{pmatrix}^* \begin{pmatrix} (r_{T+1}^c)^2 & r_{T+1}^c \bar{r}_j^* \\ r_{T+1}^c \bar{r}_j & \Sigma_j + \bar{r}_j \bar{r}_j^* \end{pmatrix} \begin{pmatrix} x_j^c \\ x_j \end{pmatrix}$$

unter

$$x_j^c = r_t^c x_{\pi(j)}^c - (e+c)^* v_j + (e-d)^* w_j + \phi_t \qquad j \in V$$

$$x_j = \mathrm{Diag}(r_j) x_{\pi(j)} + v_j - w_j \qquad j \in V$$

$$\sum_{j\in L} p_j (r_{T+1}^c x_j^c + \bar{r}_j^* x_j) = \rho$$

$$u_j \geq 0, \quad x_j \in [x_j^{\min}, x_j^{\max}], \quad B_j x_j \geq 0 \qquad j \in V$$

Special Property

- Investment reduction below 100% possible
- Equivalent: 100% investment with lower semi-variance as risk measure

Application Problem

Strategic Portfolio Management (Pension Fund)

- Quarterly rebalancing
- Tests 01/1999; routine application 10/1999; commercial software with GUI 2003

Portfolio (12 assets + cash)

- 4 bonds: CHF, EUR, USD, GBP
- 8 stocks: Switzerland, France, Germany, UK, Netherlands,
  North America, Japan, Emerging Markets

Typical Problem

- 27,225 scenarios, 767,000 variables, 280 MB
- PC 2.4 GHz: 2:00 min, 69 iterations

Large Problem

- 53,235 scenarios, 1,971,000 variables, 854 MB
- PC 2.4 GHz: 16:20 min, 187 iterations

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Current Work

DREWAG Model

Refs: Eichhorn, Römisch, Wegner-Specht 2005 (2×)

- Mean-risk optimization of electricity portfolios
- Municipal power utility (price taker)
- Combined heat and power production facility
- Electricity futures, spot market, supply contracts (EEX)
- Uncertain electricity & heat demands, spot & future prices
- Horizon = 1 year with hourly discretization
- Polyhedral multiperiod risk measures
- Four scenario trees: input tree, future tree, trading day tree, contract tree
  various trading windows and delays of decisions
  (day ahead trading . . . ⇒ nonstandard nonanticipativity constraints)

Current Work

Upcoming Project: VKW Model

Refs: Steinberger 2004; Steinberger, Zinner 2004

- Related planning problem of Vorarlberger Kraftwerke, Austria
- Hydroelectric plants and pump storage plants
- Production = 11% of demand, trading = 89% (Illwerke, ENBW, EEX)
- Horizon 1–2 years, discretization 1 hour or 15 minutes
- Deterministic LP model: up to 2 GB under CPLEX
- Crucial in SP algorithm:
  Data management, storage efficiency, structure exploitation
- Theoretical framework:
  Generalized nonanticipativity & embedding into single scenario tree [St. 2001]

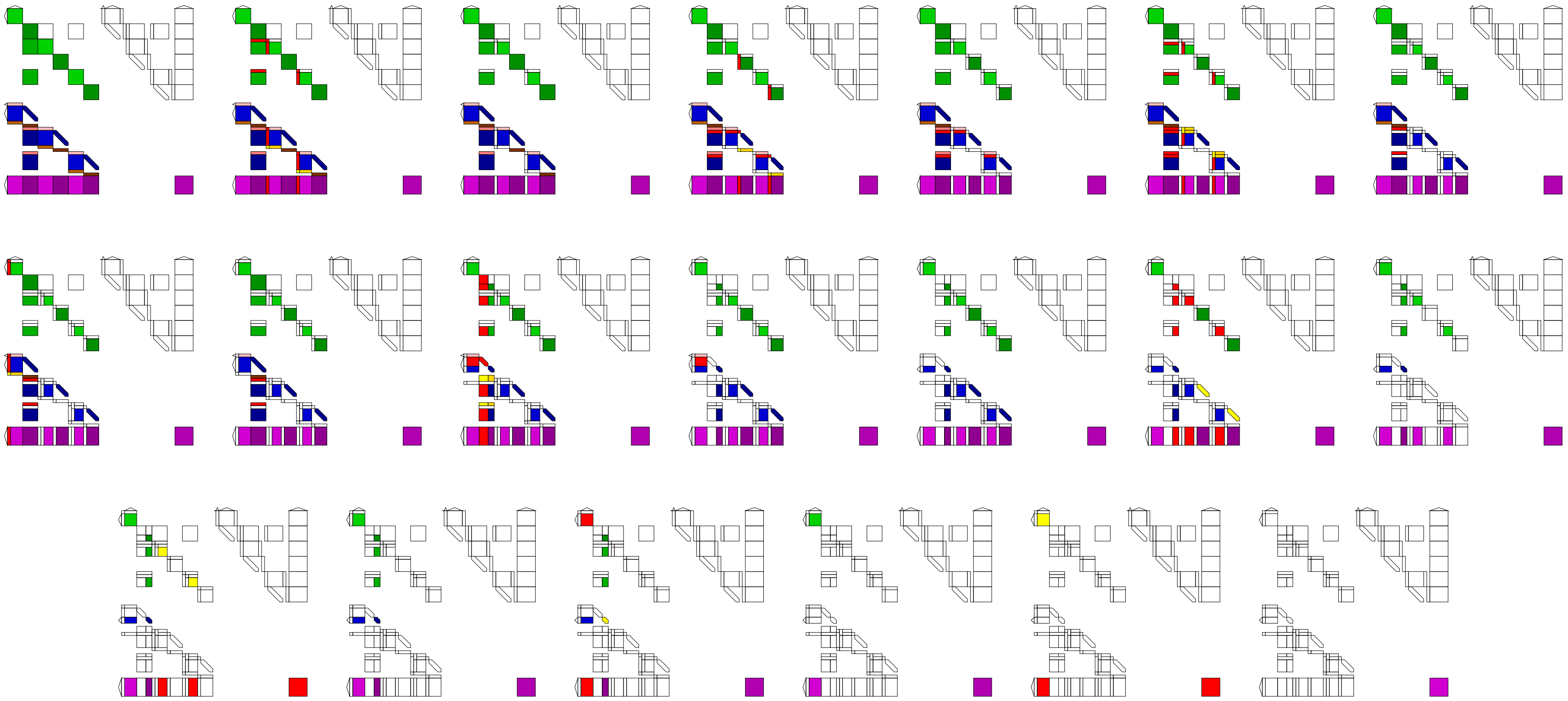
Conclusions

- Robust and efficient solution method for nonlinear stochastic programs
  - global scenario coupling ($\longrightarrow$ mean-risk)
  - arbitrary nonanticipativity constraints
- Natural classification of restrictions
- Natural block level KKT faktorization
  - Fixed elimination scheme (block pivots)
  - Linear complexity (storage, runtime)
- Exploitation of problem specific sub-structure: implementation
  - Static fill-in analysis
  - Automatic generation of custom sparse solvers (Huțanu 2002)
- Strategic Portfolio Management
  - Custom code with full structure exploitation
  - Practical application since 1999
  - Commercial software with GUI since 2003

Conclusions

KKT Matrix Factorization

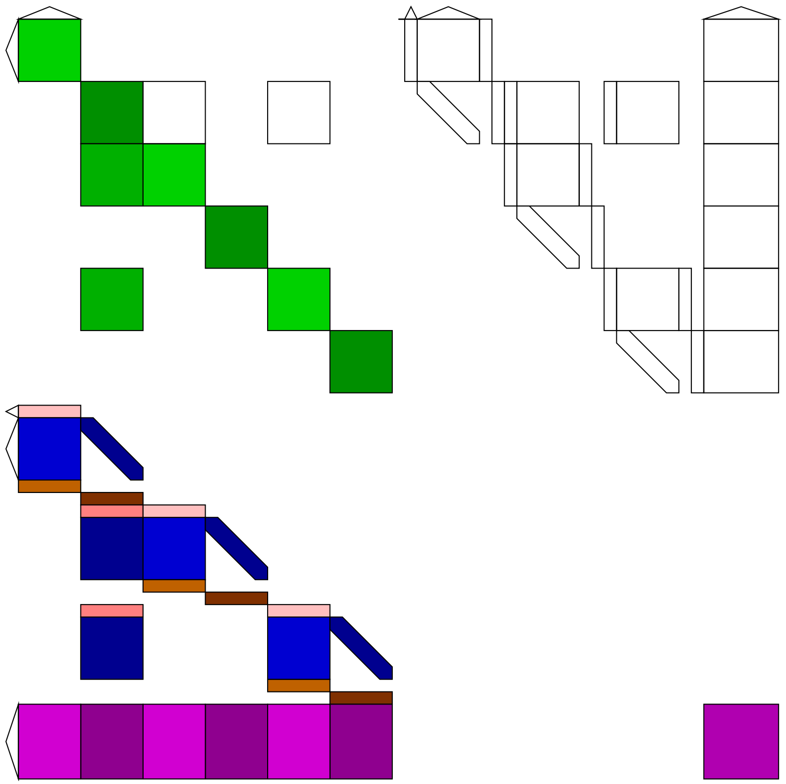
Fine Block Structure



Conclusions

KKT Matrix Factorization

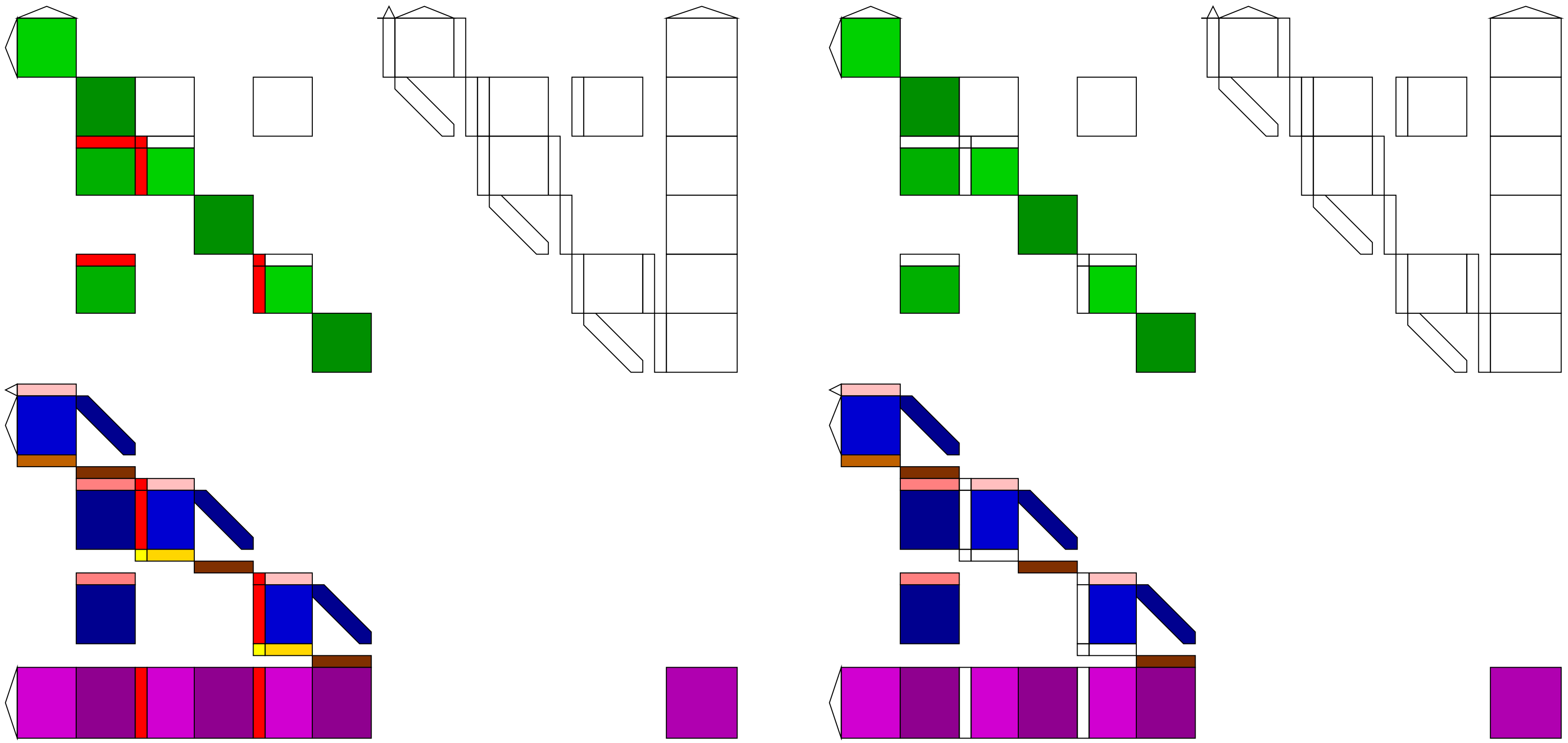
Fine Block Structure



Conclusions

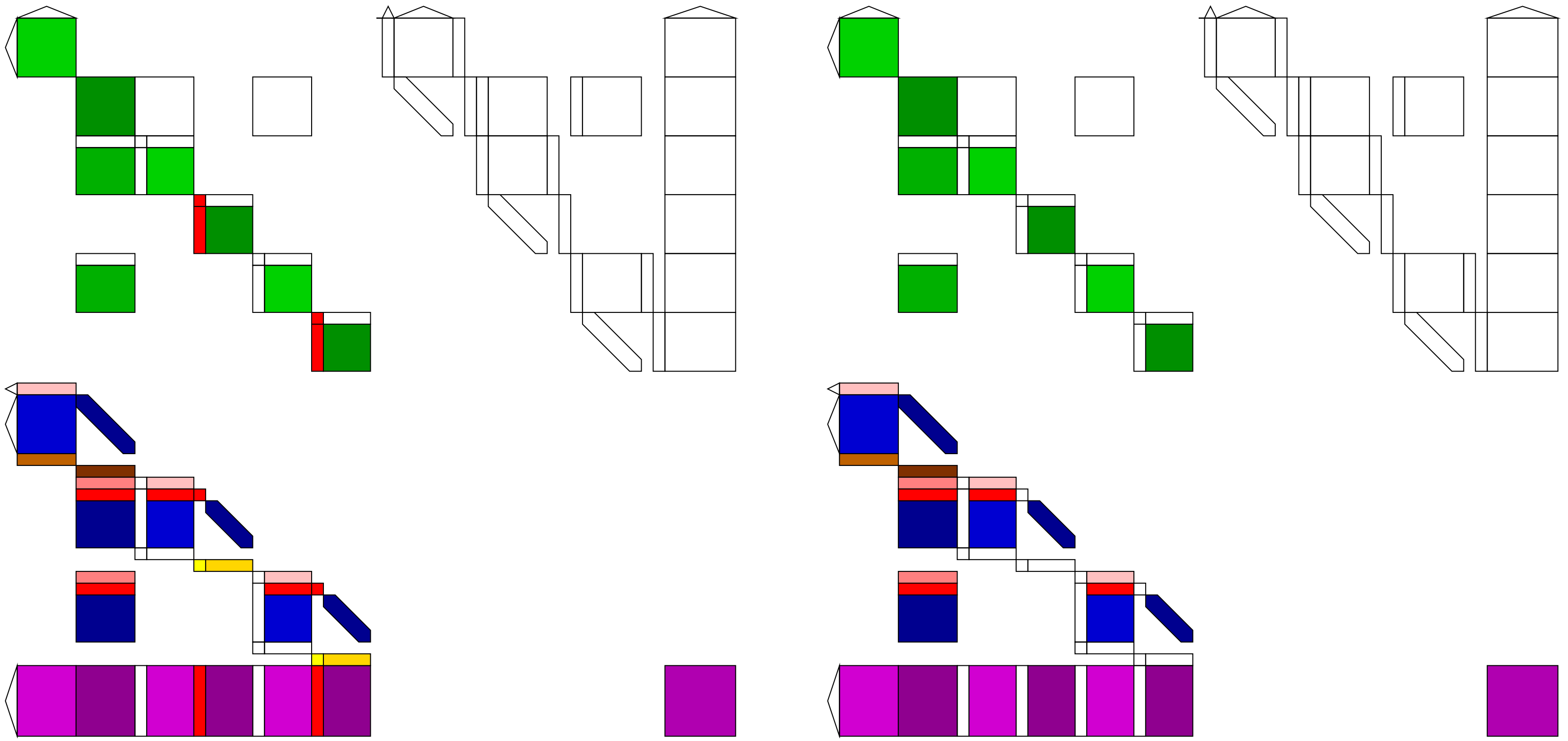
KKT Matrix Factorization

Projection: Control Constraints (children)



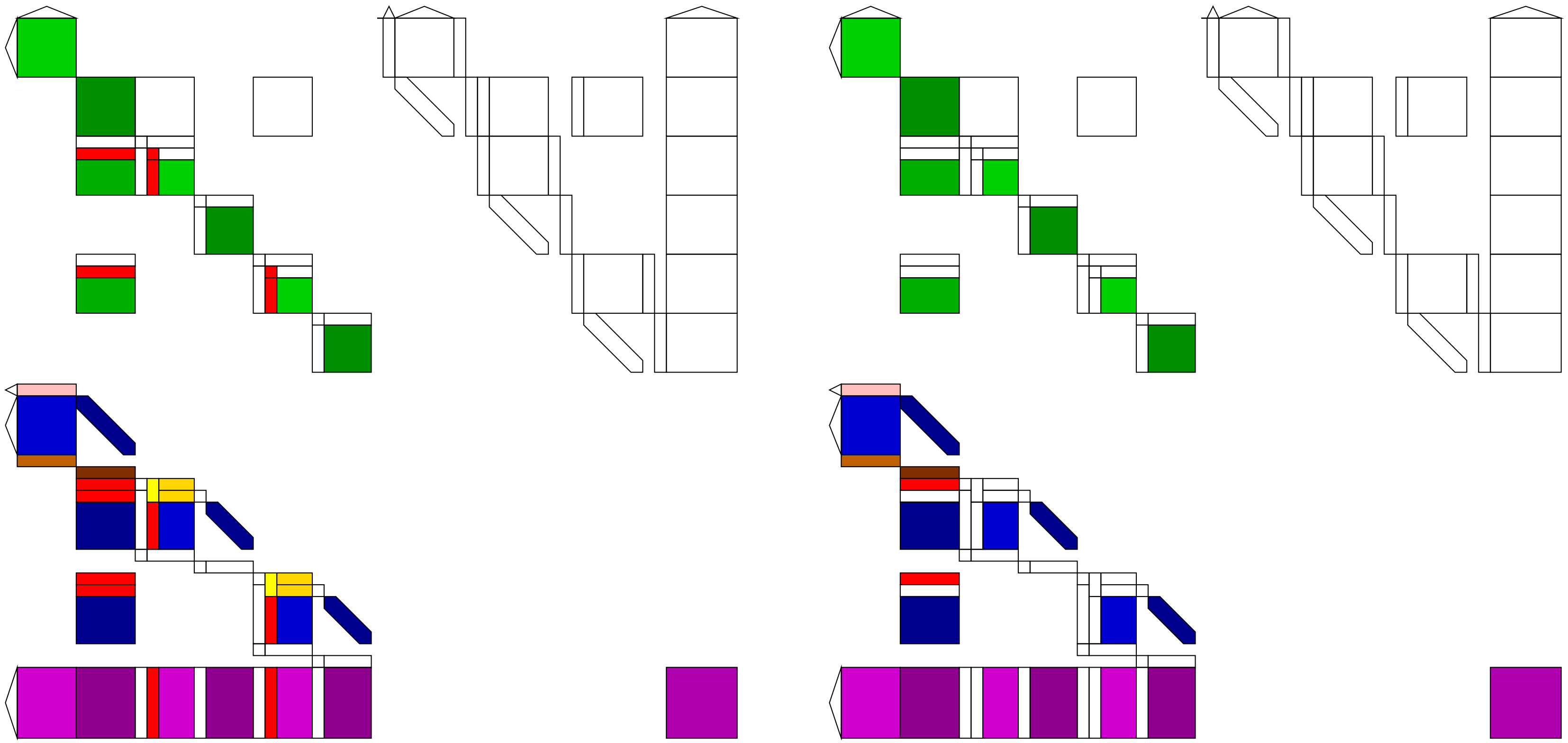
KKT Matrix Factorization

Projection: State Constraints (children)



KKT Matrix Factorization

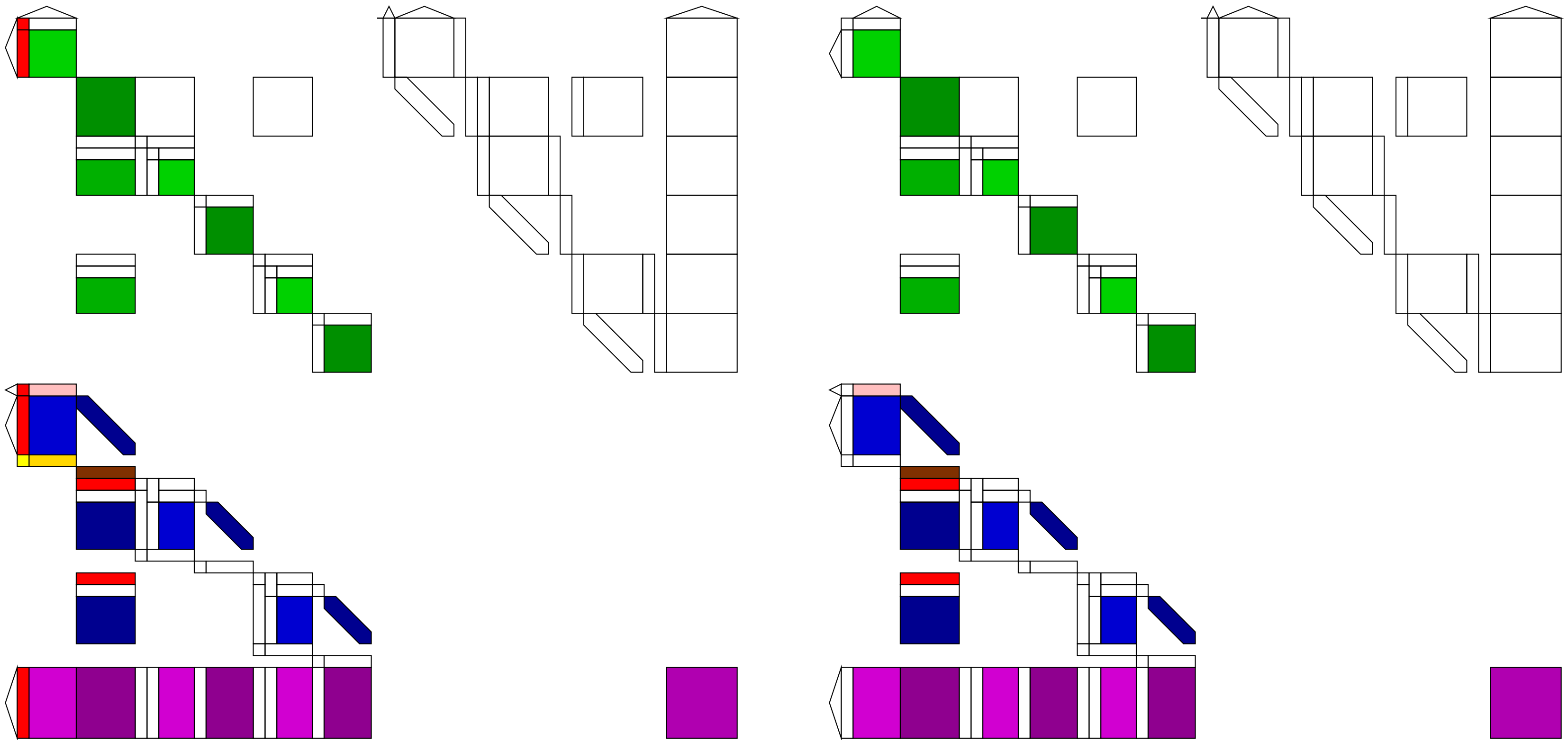
Projection: Combined Mixed Constraints (children)



Conclusions

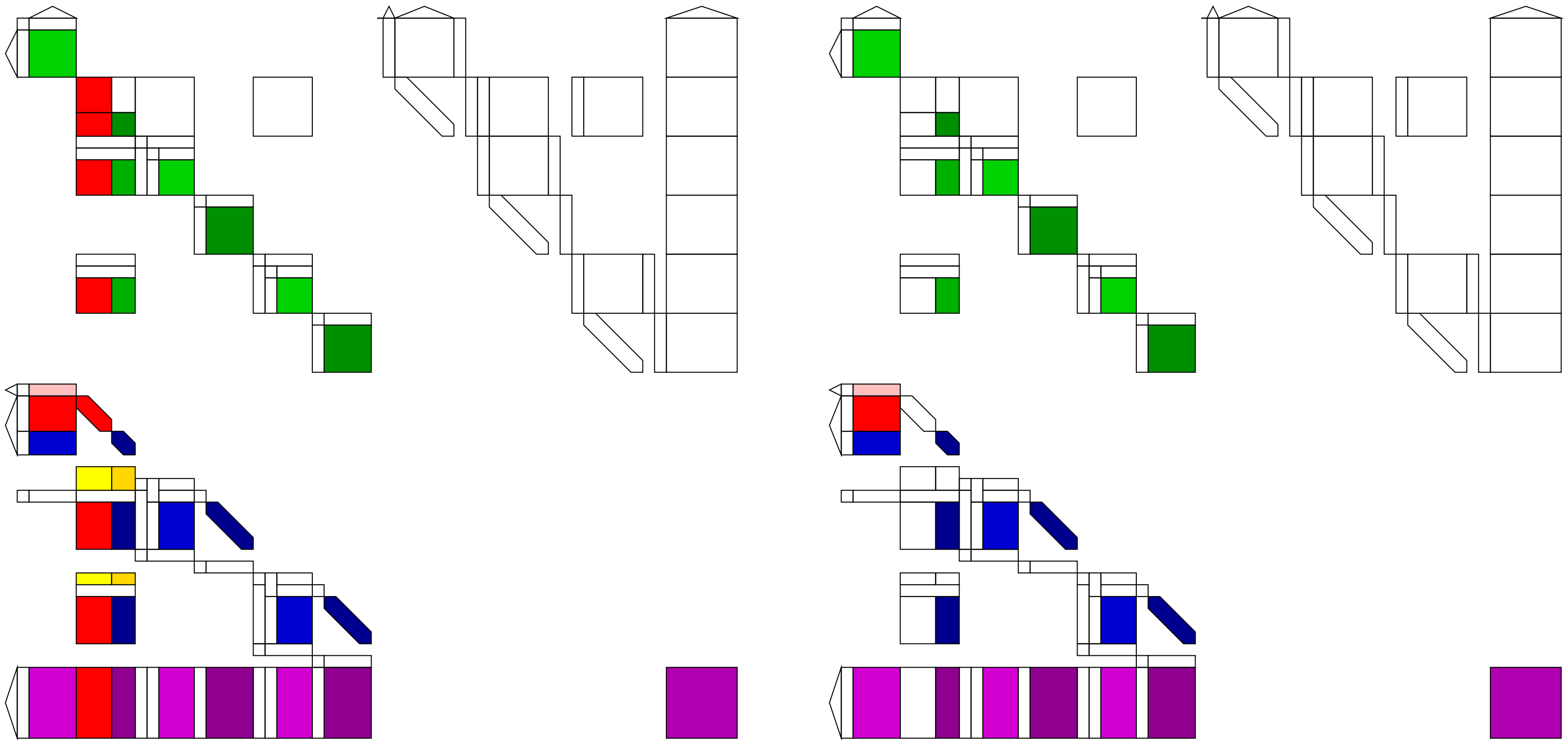
KKT Matrix Factorization

Projection: Control Constraints (parent)



KKT Matrix Factorization

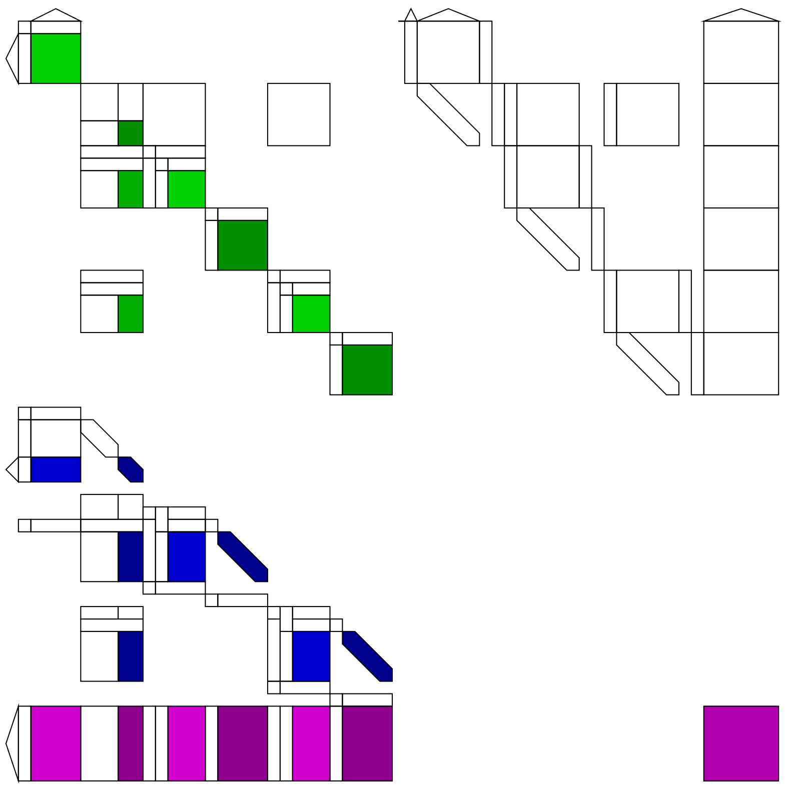
Projection: Combined State Constraints (parent)



Conclusions

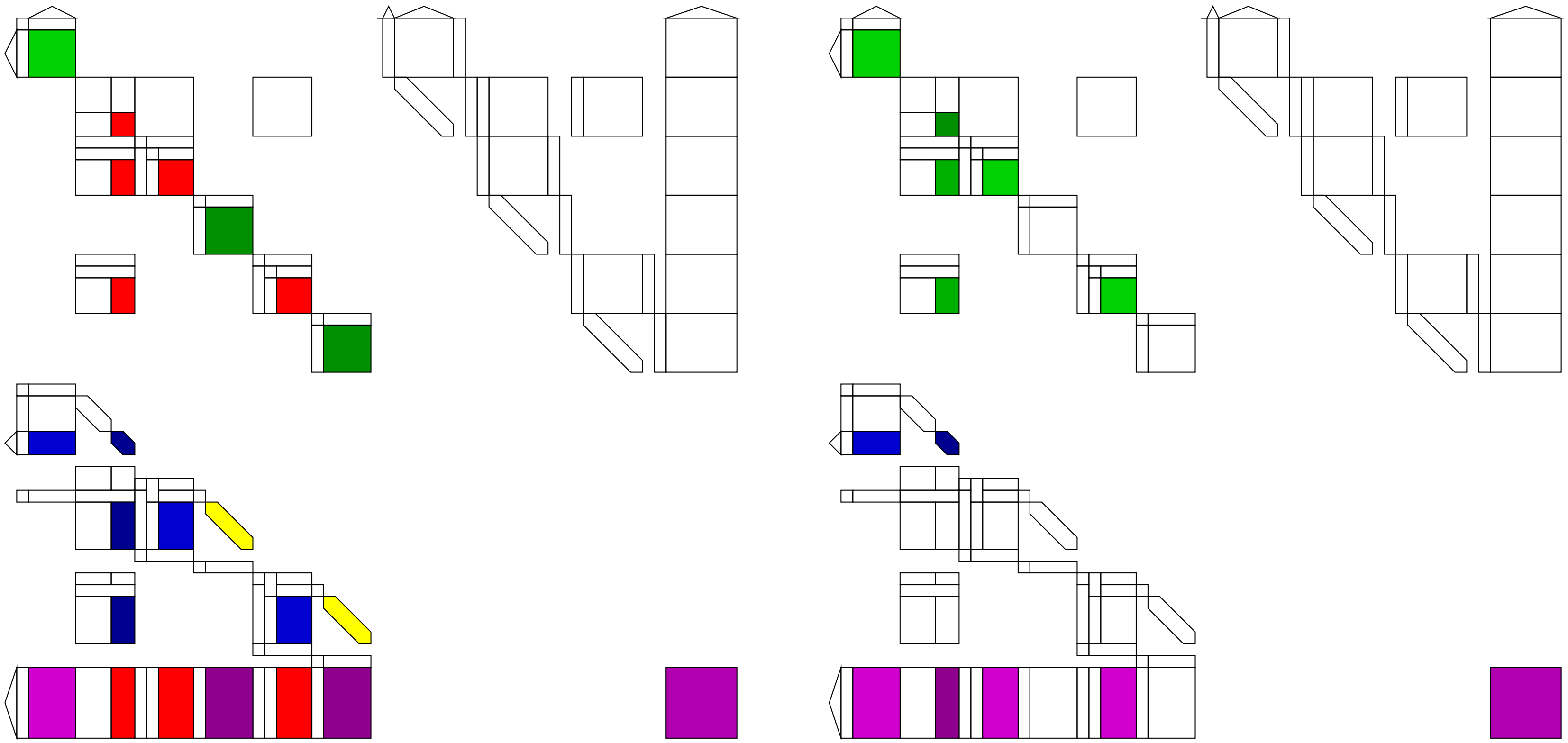
KKT Matrix Factorization

Projection: Combined Mixed Constraints (parent)



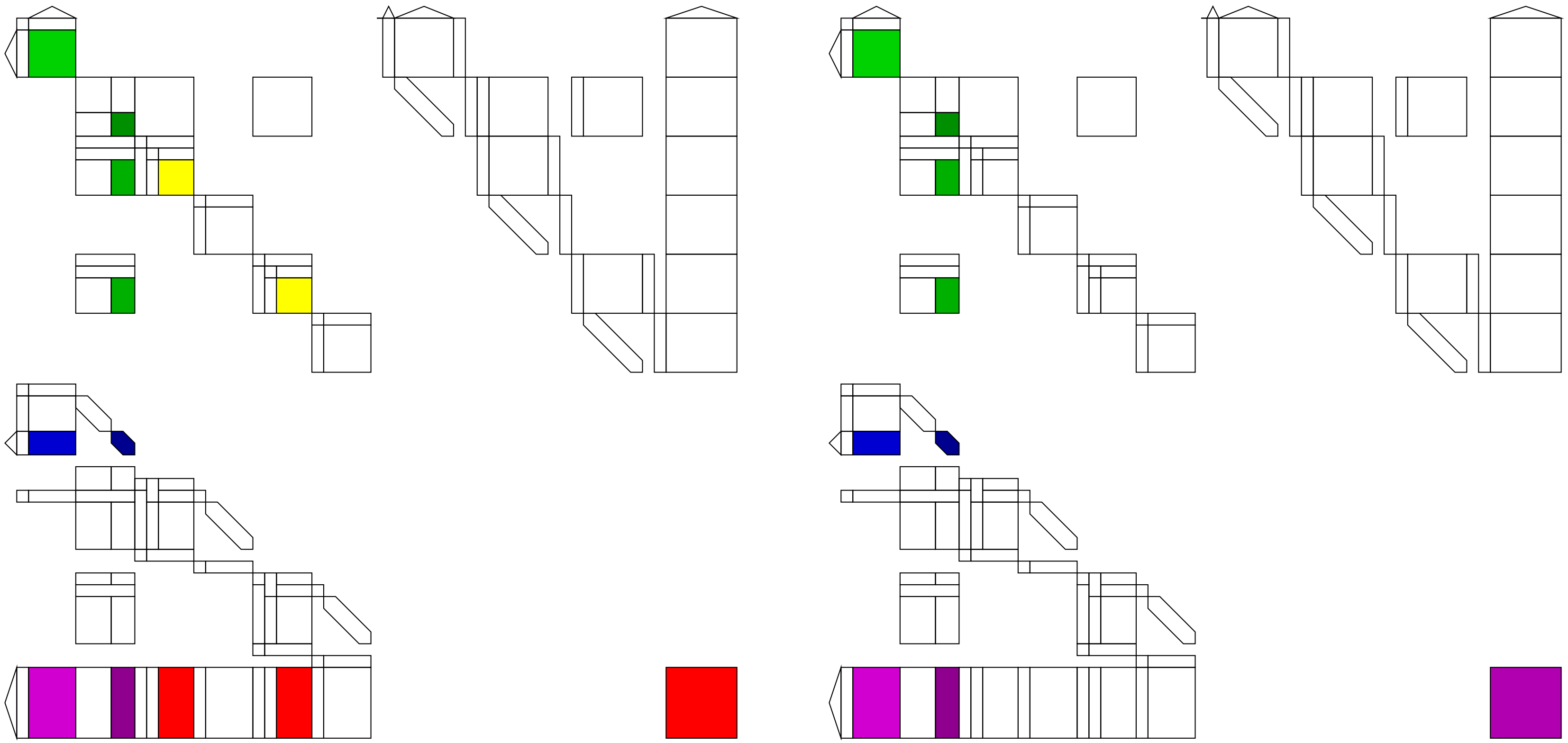
KKT Matrix Factorization

Projection: Dynamics (children)



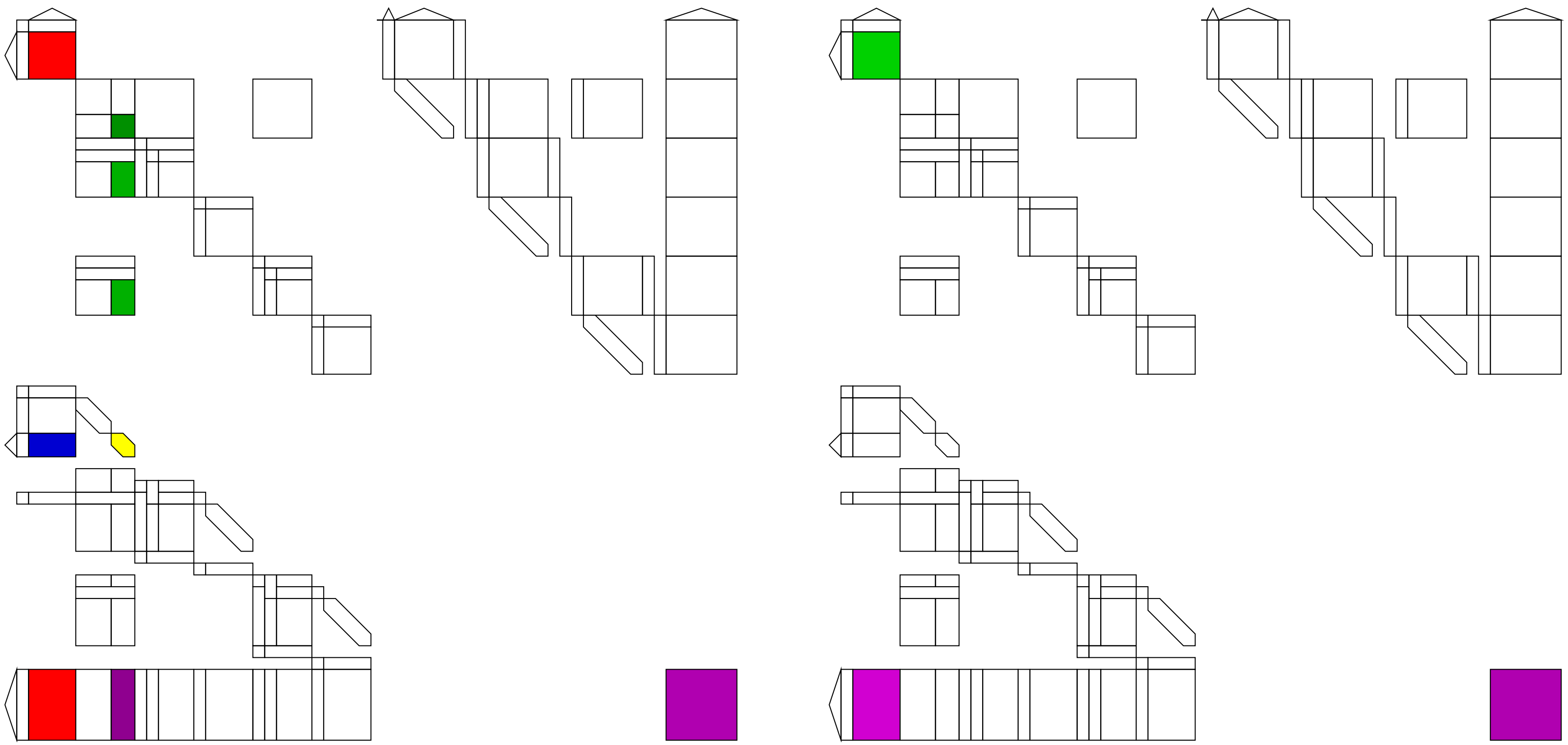
KKT Matrix Factorization

Recursion: Control (children)



KKT Matrix Factorization

DP Recursion: Dynamics (parent)



Conclusions

KKT Matrix Factorization

DP Recursion: Control (parent)