Weierstraß-Institut für Angewandte Analysis und Stochastik

BASF Ludwigshafen

John Schoenmakers; joint with C. Bender and A. Kolodko

Iterative methods for complex structured callable products

Leibniz Gemeinschaft

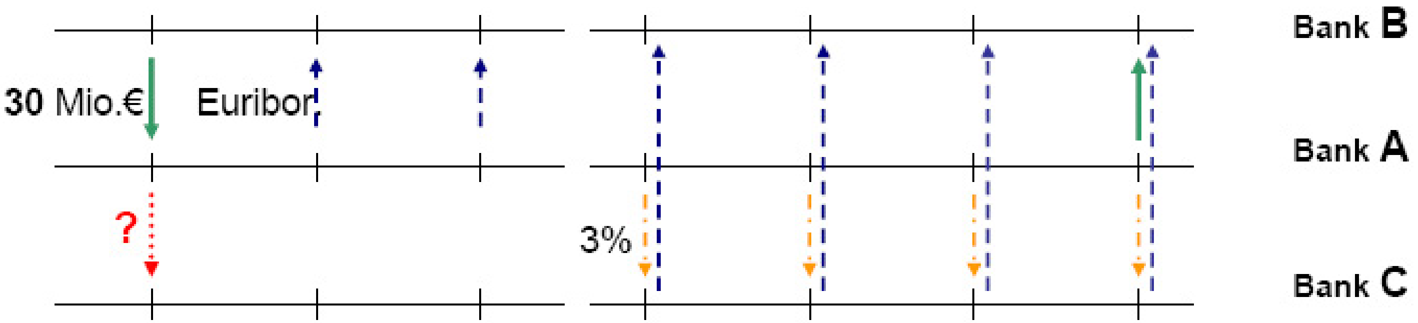
DFG Research Center MATHEON
Mathematics for key technologies

Simple example: (Bermudan) callable interest rate swap

Euribor: Interest rate for a loan between banks

Contract I: A borrows from B 30 Mio. € over a period of 10 years and pays quarterly the 3M-Euribor.

Contract II: A buys from C a Bermudan swaption, i.e. the right to choose a payment date of contract I, from which on C pays quarterly the 3M-Euribor to B and receives a fixed payment of 3% from A.



‘Exotic’ example: cancelable snowball swap

Snowball swap: Instead of the floating spot rate the holder pays a starting coupon rate $\mathbf{I}$ over the first year and in the forthcoming years

$$\max(\mathbf{K} + \text{previous coupon} - \text{spot rate}, 0),$$

where the first coupon $\mathbf{I}$ and the strike rate $\mathbf{K}$ are specified in the contract.

Cancelable snowball swap: The holder has the right to cancel this contract.

What is the fair value of this cancelable product?

▷ **Mathematical problem:**

Optimal stopping (calling) of a reward (cash-flow) process Z depending an underlying (interest rate) process L

▷ **Typical difficulties:**

– L is usually high dimensional, for Libor interest rate models, $d = 10$ and up, so PDE methods do not work in general

– Z may only be virtually known, e.g. $Z_i = E^{\mathcal{F}_i} \sum_{j \geq i} C(L_j)$ for some pay-off function C, rather than simply $Z_i = C(L_i)$

– Z may be path-dependent

▷ **Approach:**

– Newly developed path-by-path policy iteration methodology

– Solution via efficient and numerically stable Monte Carlo algorithms

The standard Bermudan pricing problem

Consider an underlying process L in $\mathbb{R}^D$, e.g. a system of asset prices or Libor rates and a set of (future) dates $\mathbb{T} := \{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k\}$

Bermudan derivative: An option to exercise a cashflow $C(\mathcal{T}_\tau, L(\mathcal{T}_\tau))$ at a future time $\mathcal{T}_\tau \in \mathbb{T}$, to be decided by the option holder

Valuation: If N, with $N(0) = 1$, is some discounting numeraire and P the with N associated pricing measure, then with $Z_\tau := C(\mathcal{T}_\tau, L(\mathcal{T}_\tau))/N(\mathcal{T}_\tau)$, the $t = 0$ price of the option is given by the optimal stopping problem

$$V_0 = \sup_{\tau \in \{0,\ldots,k\}} E^{\mathcal{F}_0} Z^{(\tau)},$$

where the supremum runs over all stopping indexes τ with respect to $\{\mathcal{F}_{\mathcal{T}_i}, 0 \le i \le k\}$, where $(\mathcal{F}_t)_{t\ge 0}$ is the usual filtration generated by L.

At a future time point t, when the option is not exercised before t, the Bermudan option value is given by

$$V_t = N(t) \sup_{\tau \in \{\kappa(t),\ldots,k\}} E^{\mathcal{F}_t} Z^{(\tau)}$$

with $\kappa(t) := \min\{m : \mathcal{T}_m \geq t\}$.

The process

$$Y_t^* := \frac{V_t}{N(t)},$$

called the *Snell-envelope* process, is a supermartingale, i.e.

$$E^{\mathcal{F}_s} Y_t^* \leq Y_s^*$$

Canonical Solution by Backward Dynamic Programming

Set $Y^{*(i)} := Y^*(\mathcal{T}_i), \quad \mathcal{F}^{(i)} := \mathcal{F}_{\mathcal{T}_i}$. At the last exercise date $\mathcal{T}_k$ we have,

$$Y^{*(k)} = Z^{(k)}$$

and for $0 \leq j < k$,

$$Y^{*(j)} = \max\left(Z^{(j)}, E^{\mathcal{F}_j} Y^{*(j+1)}\right).$$

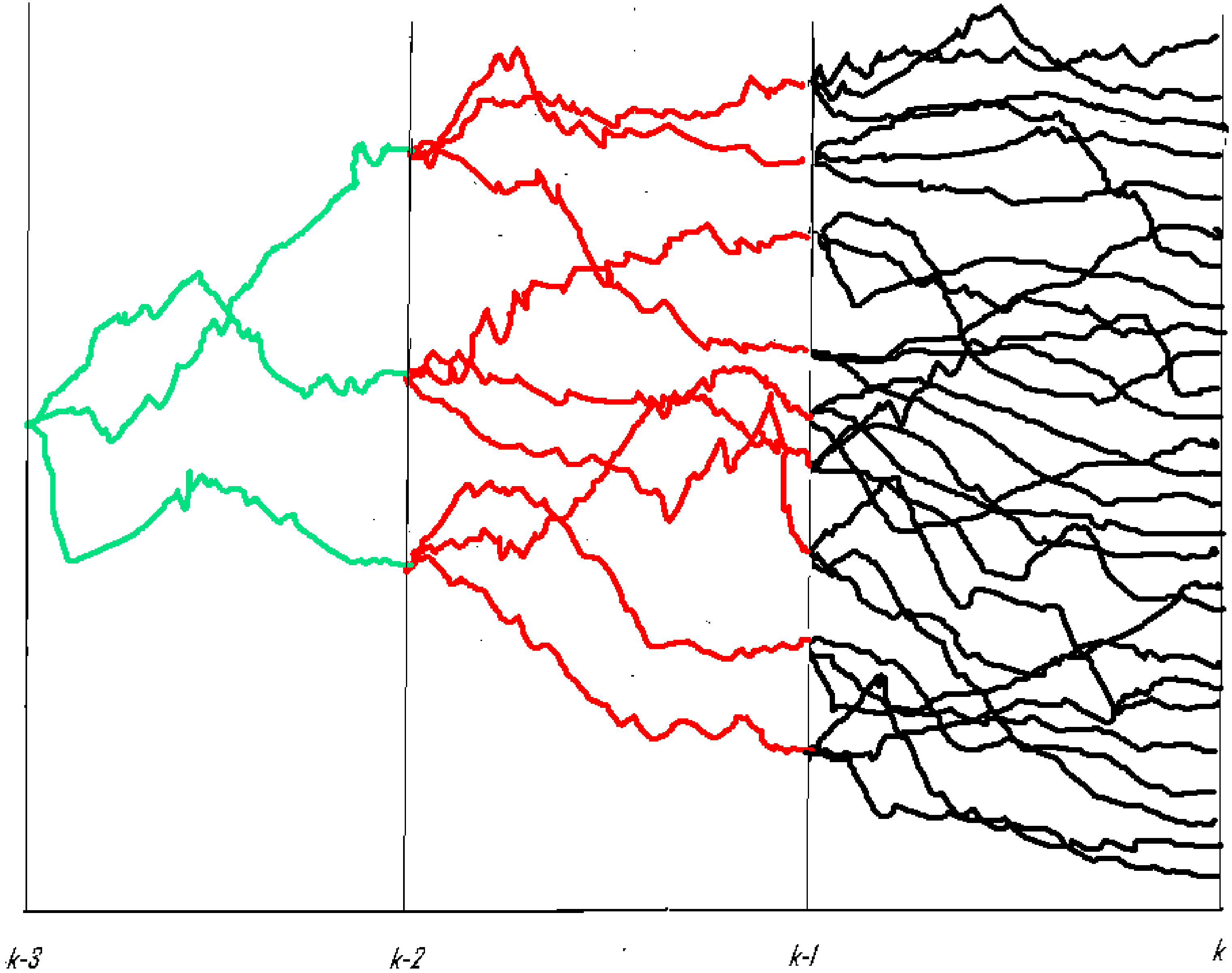
The first optimal stopping time (index) is then obtained by

$$\tau_i^* = \inf\left\{j,\ i \leq j \leq k : Y^{*(j)} \leq Z^{(j)}\right\}.$$

$\longrightarrow$ Nested Monte Carlo simulation of the price Y_0^* would thus require N^k samples when conditional expectations are computed with N samples

Typically, N=10000, k=10 exercise opportunities, give 10^{40} samples !!

Iterative methods for complex structured callable products



Improving upon given input stopping policies

We consider an input stopping family (policy) (τ_i), which satisfies the consistency conditions:

$$i \le \tau_i \le k,\ \tau_k = k, \quad \tau_i > i \Rightarrow \tau_i = \tau_{i+1}, \qquad 0 \le i < k,$$

and the corresponding lower bound process Y for the Snell envelope Y^*,

$$Y^{(i)} := E^{\mathcal{F}^{(i)}} Z^{(\tau_i)} \le Y^{*(i)}$$

Example input policies:

- ▷ The policy, $\tau_i \equiv i$. says: exercise immediately!
- ▷ The policy $\tau_i := \inf\{j \ge i : L(\mathcal{T}_j) \in G \subset \mathbb{R}^D\}$ exercises when the underlying process L enters a certain region G
- ▷ The policy $\tau_i = \inf\{j : i \le j \le k,\ \max_{p:\, j \le p \le k} E^{\mathcal{F}^{(j)}} Z^{(p)} \le Z^{(j)}\}$ waits until the cashflow is at least equal to the maximum of still-alive Europeans ahead

One step improvement:

Introduce an intermediate process

$$\widetilde{Y}^{(i)} := \max_{p:\, i \le p \le k} E^{\mathcal{F}^{(i)}} Z^{(\tau_p)}$$

and use $\widetilde{Y}^{(i)}$ as a new exercise criterion to define a new exercise policy

$$\begin{aligned} \widehat{\tau}_i : &= \inf\{j : i \le j \le k,\ \widetilde{Y}^{(j)} \le Z^{(j)}\} \\ &= \inf\{j : i \le j \le k,\ \max_{p:\, j \le p \le k} E^{\mathcal{F}^{(j)}} Z^{(\tau_p)} \le Z^{(j)}\} \quad 0 \le i \le k \end{aligned}$$

Then consider the process

$$\widehat{Y}^{(i)} := E^{\mathcal{F}^{(i)}} Z^{(\widehat{\tau}_i)}$$

as a next approximation of the Snell envelope

Key Proposition (Kolodko & Schoenmakers 2006) It holds

$$Y^{(i)} \le \widetilde{Y}^{(i)} \le \widehat{Y}^{(i)} \le Y^{*(i)}, \quad 0 \le i \le k$$

Iterative construction of the optimal stopping time

Take an initial family of stopping times $(\tau_i^{(0)})$ satisfying the consistency conditions

$$i \leq \tau_i^{(0)} \leq k, \quad \tau_k^{(0)} = k, \quad \tau_i > i \Rightarrow \tau_i = \tau_{i+1},$$

and set $Y^{0(i)} := E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{(0)})}$, $0 \leq i \leq k$. Suppose that for $m \geq 0$ the pair

$$\left((\tau_i^{(m)}), (Y^{m(i)}) \right)$$

is constructed with $\tau_i^{(m)}$ being consistent and $Y^{m(i)} := E^{\mathcal{F}_i} Z^{(\tau_i^{(m)})}, \quad 0 \leq i \leq k$. Then define

$$\begin{aligned} \tau_i^{(m+1)} &:= \inf\{j : i \leq j \leq k, \max_{p:\, j \leq p \leq k} E^{\mathcal{F}^{(j)}} Z^{(\tau_p^{(m)})} \leq Z^{(j)}\} \\ &=: \inf\{j : i \leq j \leq k, \widetilde{Y}^{m+1(j)} \leq Z^{(j)}\}, \quad 0 \leq i \leq k, \end{aligned}$$

and set

$$Y^{m+1(i)} := E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{(m+1)})}$$

By the ‘key proposition’ we thus have

$$Y^{0(i)} \leq Y^{m(i)} \leq \widetilde{Y}^{m+1(i)} \leq Y^{m+1(i)} \leq Y^{*(i)}, \quad 0 \leq m < \infty, \quad 0 \leq i \leq k.$$

and it is shown that for $m \geq 1$,

$$\tau_i^{(m)} \leq \tau_i^{(m+1)} \leq \tau_i^*,$$

where τ_i^* is the first optimal stopping time.

We so may take limits and it holds,

$$Y^{\infty(i)} := \text{(a.s.)} \lim_{m\uparrow\infty} \uparrow Y^{m(i)} \quad \text{and} \quad \tau_i^{\infty} := \text{(a.s.)} \lim_{m\uparrow\infty} \uparrow \tau_i^{(m)}, \quad 0 \leq i \leq k, \text{ and,}$$

$$Y^{\infty(i)} = \text{(a.s.)} \lim_{m\uparrow\infty} \uparrow E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{(m)})} = E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{\infty})}, \quad 0 \leq i \leq k$$

Theorem

The constructed limit process Y^{∞} coincides with the Snell envelope process Y^{*} and (τ_i^{∞}) coincides with (τ_i^{*}); the family of first optimal stopping times. We have

$$Y^{*(i)} = Y^{\infty(i)} = E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{\infty})}, \quad 0 \leq i \leq k.$$

Moreover: It even holds (see also Bender & Schoenmakers 2004)

$$Y^{m(i)} = Y^{*(i)} \quad \text{for} \quad m \geq k - i$$

$\longrightarrow$ After $k = \#$exercise dates iterations the Snell Envelope is attained!

Iteration procedure vs backward dynamic program

	— *Exercise* *date* $\rightarrow$					
	0	1	$\cdots$	$k-2$	$k-1$	k
0	$Y_0^{(0)}$	$Y_1^{(0)}$	$\cdots$	$Y_{k-2}^{(0)}$	$Y_{k-1}^{(0)}$	Y_k^*
1	$Y_0^{(1)}$	$Y_1^{(1)}$	$\cdots$	$Y_{k-2}^{(1)}$	Y_{k-1}^*	Y_k^*
| *Iteration level* $\downarrow$ 2	$Y_0^{(2)}$	$Y_1^{(2)}$		Y_{k-2}^*	Y_{k-1}^*	Y_k^*
$\cdot$	$\cdot$	$\cdot$		$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$		$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$		$\cdot$	$\cdot$	$\cdot$
$k-1$	$Y_0^{(k-1)}$	Y_1^*	$\cdots$	Y_{k-2}^*	Y_{k-1}^*	Y_k^*
k	Y_0^*	Y_1^*	$\cdots$	Y_{k-2}^*	Y_{k-1}^*	Y_k^*

Upper approximations of the Snell envelope by Duality

The Dual Method

Consider a discrete martingale $(M_j)_{j=0,...,k}$ with $M_0 = 0$ with respect to the filtration $\left(\mathcal{F}^{(j)}\right)_{j=0,...,k}$. Following Rogers (2001) we observe that

$$\begin{aligned} Y_0 &= \sup_{\tau\in\{0,...,k\}} E^{\mathcal{F}_0} Z^{(\tau)} = \sup_{\tau\in\{0,,...,k\}} E^{\mathcal{F}_0}\left[Z^{(\tau)} - M_\tau\right] \\ &\leq E^{\mathcal{F}_0} \max_{0\leq j\leq k}\left[Z^{(j)} - M_j\right] \end{aligned}$$

Hence the r.h.s. gives an upper bound for the Bermudan price $V_0 = Y_0$.

Theorem (Davis Karatzas (1994), Rogers (2001), Haugh & Kogan (2001))

Let M^* be the (unique) Doob-Meyer martingale part of $\left(Y^{*(j)}\right)_{0\le j\le k}$, i.e. M^* is an $\left(\mathcal{F}^{(j)}\right)$-martingale which satisfies

$$Y^{*(j)} = Y_0^* + M_j^* - F_j^*, \quad j = 0, ..., k,$$

with $M_0^* := F_0^* := 0$ and F^* being such that F_j^* is $\mathcal{F}^{(j-1)}$ measurable for $j = 1, ..., k$. Then we have

$$Y_0^* = E^{\mathcal{F}_0} \max_{0\le j\le k} \left[Z^{(j)} - M_j^*\right].$$

Convergent upper bounds from a convergent sequence of lower bounds

From our previously constructed sequence of lower bound processes $Y^{m(i)}$ with $Y^{m(i)} \uparrow Y^{*(i)}$, we deduce by duality a sequence of upper bound processes:

$$Y_{up}^{m(i)} := E^{\mathcal{F}_i} \max_{i \le j \le k} \left(Z^{(j)} - \sum_{l=i+1}^{j} Y^{m(l)} + \sum_{l=i+1}^{j} E^{\mathcal{F}^{(l-1)}} Y^{m(l)} \right) =: Y^{m(i)} + \Delta^{m(i)}.$$

Then, by a theorem of (Kolodko & Schoenmakers 2004),

$$0 \le \Delta^{m(i)} \le E^{\mathcal{F}_i} \sum_{j=i}^{k-1} \max \left(E^{\mathcal{F}_j} Y^{m(j+1)} - Y^{m(j)}, \ 0 \right).$$

Thus, by letting $m \uparrow \infty$ on the r.h.s., (a.s.) $\lim_{m \to \infty} \Delta^{m(i)} = 0, \quad 0 \le i \le k$. Hence, the sequence Y_{up}^{m} converges to the Snell envelope also, i.e.,

$$\text{(a.s.)} \lim_{m \to \infty} Y_{up}^{m(i)} = \text{(a.s.)} \lim_{m \to \infty} Y^{m(i)} = Y^{*(i)}, \quad 0 \le i \le k.$$

A numerical example: Bermudan swaptions in the LIBOR market model

Consider the Libor Market Model with respect to a tenor structure $0 < T_1 < T_2 < \ldots < T_n$, e.g. in the spot Libor measure P^* induced by the numeraire

$$B^*(t) := \frac{B_{m(t)}(t)}{B_1(0)} \prod_{i=0}^{m(t)-1} (1 + \delta_i L_i(T_i))$$

with $m(t) := \min\{m : T_m \geq t\}$.
The dynamics of the forward Libor $L_i(t)$ is given by a system of SDE's

$$dL_i = \sum_{j=m(t)}^{i} \frac{\delta_j L_i L_j \, \gamma_i \cdot \gamma_j}{1 + \delta_j L_j} dt + L_i \, \gamma_i \cdot dW^*.$$

Here $\delta_i = T_{i+1} - T_i$ are day count fractions, and

$$t \to \gamma_i(t) = (\gamma_{i,1}(t), \ldots, \gamma_{i,d}(t))$$

are deterministic volatility vector functions defined in $[0, T_i]$, called factor loadings.

A (payer) Swaption over a period $[T_i, T_n]$, $1 \leq i \leq k$. A *swaption contract* with maturity T_i and strike θ with principal $\$1$ gives the right to contract at T_i for paying a fixed coupon θ and receiving floating Libor at the settlement dates T_{i+1},...,T_n. So by this definition, its cashflow at maturity is

$$S_{i,n}(T_i) := \left(\sum_{j=i}^{n-1} B_{j+1}(T_i)\delta_j \left(L_j(T_i) - \theta\right) \right)^+ .$$

A Bermudan Swaption gives the the right to exercise a cashflow

$$C_{T_\tau} := S_{\tau,n}(T_\tau)$$

at an exercise date $T_\tau \in \{T_1, \ldots, T_n\}$ to be decided by the option holder.

10 yr. Bermudan swaption:

Comparison of Y^1, Y^2, $Y^{1,up}$ with $\tau_i^{(0)} \equiv i$ (trivial initial stopping family)

θ	d	Y^1 (SD)	Y^2 (SD)	$Y^{1,up}$ (SD)
0.08 (ITM)	1	1104.6(0.5)	1108.9(2.4)	1109.4(0.7)
	2	1098.6(0.4)	1100.5(2.4)	1103.7(0.7)
	10	1094.4(0.4)	1096.9(2.1)	1098.1(0.6)
	40	1093.6(0.4)	1096.1(2.0)	1096.6(0.6)
0.10 (ATM)	1	374.3(0.4)	381.2(1.6)	382.9(0.8)
	2	357.9(0.3)	364.4(1.5)	366.4(0.8)
	10	337.8(0.3)	343.5(1.3)	345.6(0.7)
	40	332.6(0.3)	338.7(1.2)	341.2(0.8)
0.12 (OTM)	1	119.0(0.2)	121.0(0.6)	121.3(0.4)
	2	112.7(0.2)	113.8(0.5)	114.9(0.4)
	10	100.2(0.2)	100.7(0.4)	101.5(0.3)
	40	96.5(0.2)	96.9(0.4)	97.7(0.3)

Conclusions from the tables:

- ▷ The computed lower bound Y^2, hence the second iteration, is within 1% or less (relative to the price) of the Dual upper bound $Y^{1,up}$
- ▷ Computation times may be considered low in view of the high-dimensionality of the problem!

More general Conclusions

- ▷ Computation times may be reduced further by the scenario selection method of Bender, Kolodko, Schoenmakers
- ▷ The iterative approach provides a general method for improving upon any given input stopping policy
- ▷ The proposed iterative procedure, together with its Monte Carlo implementation, may be used generically for many (also not financial) optimal stopping problems!

Outlook

Policy iteration methods for game options: instruments with **two** cancellation parties (for example convertible bonds)

Further recent results on the policy improvement methodology

▷ Bender, Kolodko, Schoenmakers (WIAS Pr. 1071, 2005): *Enhanced policy iteration for American options via scenario selection*
  – In this work a substantial speed up of the algorithm is obtained via a scenario selection theorem

▷ Bender, Kolodko, Schoenmakers (WIAS Pr. 1061, 2005): *Iterating snowballs and related path dependent callables in a multi-factor Libor model*
  – Here the method is extended to exotic structures like the 'snowball swap' involving virtual cashflows of the form $Z_i = E^{\mathcal{F}_i} \sum_{j \geq i} C(L_j)$ for some pay-off function C.

▷ Bender & Schoenmakers (WIAS preprint 991, 2004): *An iterative algorithm for multiple stopping: Convergence and stability*
Accepted for *Advances in Appl. Prob.*
  – Extension of the method to multiple exercise problems
  – Stability proof of the resulting Monte Carlo algorithms where mathematical conditional expectations are replaced by suitable estimators

Related published work:

▷ Kolodko & Schoenmakers (2006): Iterative construction of the optimal Bermudan stopping time, *Finance and Stochastics, 10, 27–49*

▷ Schoenmakers (2005): *Robust Libor Modelling and Pricing of Derivative Products*, Chapman & Hall - CRC Press

▷ Kolodko & Schoenmakers (2004): Upper bounds for Bermudan style derivatives *Monte Carlo Meth. Appl., 10, 331–343*