Optimization Problems with Linear Chance Constraints – Structure, Numerics and Applications

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- Introduction with Examples
- Structure (convexity, differentiability, existence of solutions)
- Numerics (singular normal distributions)

75. Sitzung der GOR Arbeitsgruppe

Praxis der Mathematischen Optimierung – Optimization under Uncertainty

Bad Honnef, October 20-21, 2005

Chance Constraints

system of stochastic inequalities $g(x, \xi) \leq 0$

random parameter
(meteorological data, product demands, prices, return rate)

'here-and-now'- decisions: $x \Rightarrow \xi$

deterministic reformulations:

a) expected-value constraint: $E\, g(x, \xi) \leq 0$ or $g(x, E\,\xi) \leq 0$

b) worst-case constraint: $g(x, \xi) \leq 0 \;\; \forall \xi$

c) chance-constraint: $P(g(x, \xi) \leq 0) \geq p \quad (p \in [0,1])$

Portfolio-Optimization (1)

Distribute capital K among n assets with random return.

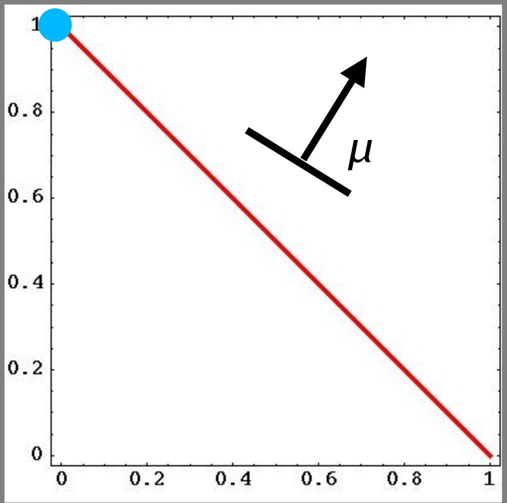
simplest model: maximize expected return!

$$x_i = \text{capital put on asset } i, \quad \xi_i = \text{return rate for asset } i$$

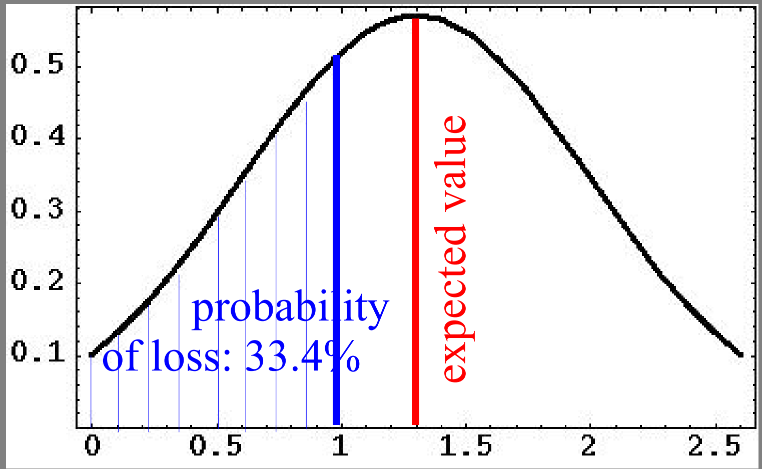
$$\xi \text{ random vector, assumption: } \xi \sim N(\mu, \Sigma) \text{ (normally distributed)}$$

➡ linear optimization problem: $\max\{\langle \mu, x\rangle \mid \sum_{i=1}^{n} x_i = K, \; x_i \geq 0\}$

Example: $\mu = (1.2, 1.3), \; \Sigma = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix}$



Distribution of Return

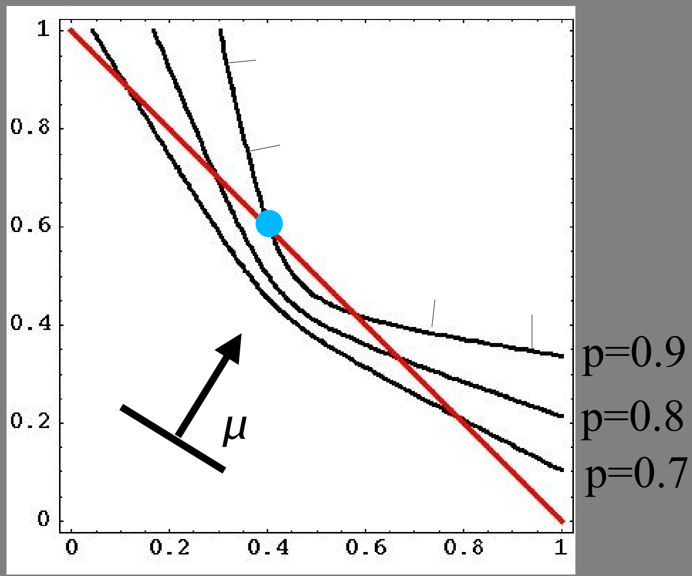


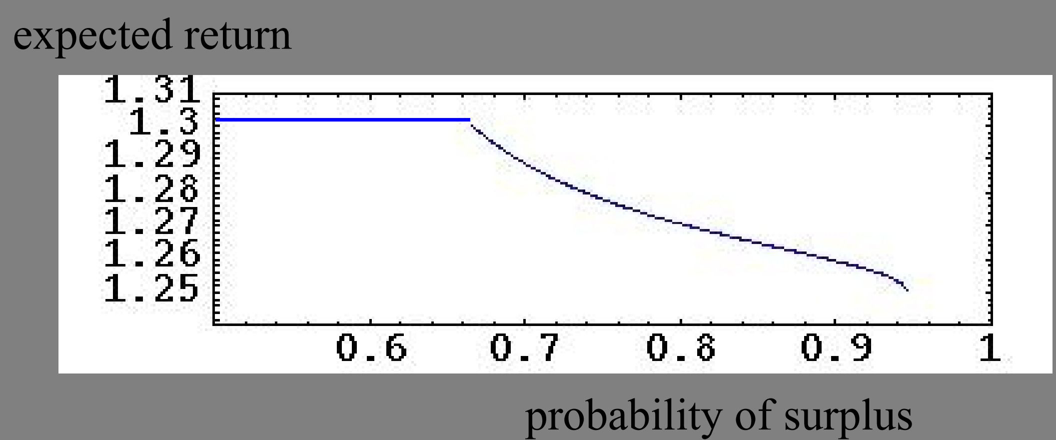
Portfolio-Optimization (2)

random return: $\langle \xi, x \rangle$ probability of surplus: $P(\langle \xi, x \rangle \geq K)$

optimization problem with chance constraints:

$$max\{\langle \mu, x \rangle \mid \sum_{i=1}^{n} x_i = K,\ \ x_i \geq 0,\ \ P(\langle \xi, x \rangle \geq K) \geq p\} \quad (p \in [0,1])$$





A Problem from Chemical Engineering

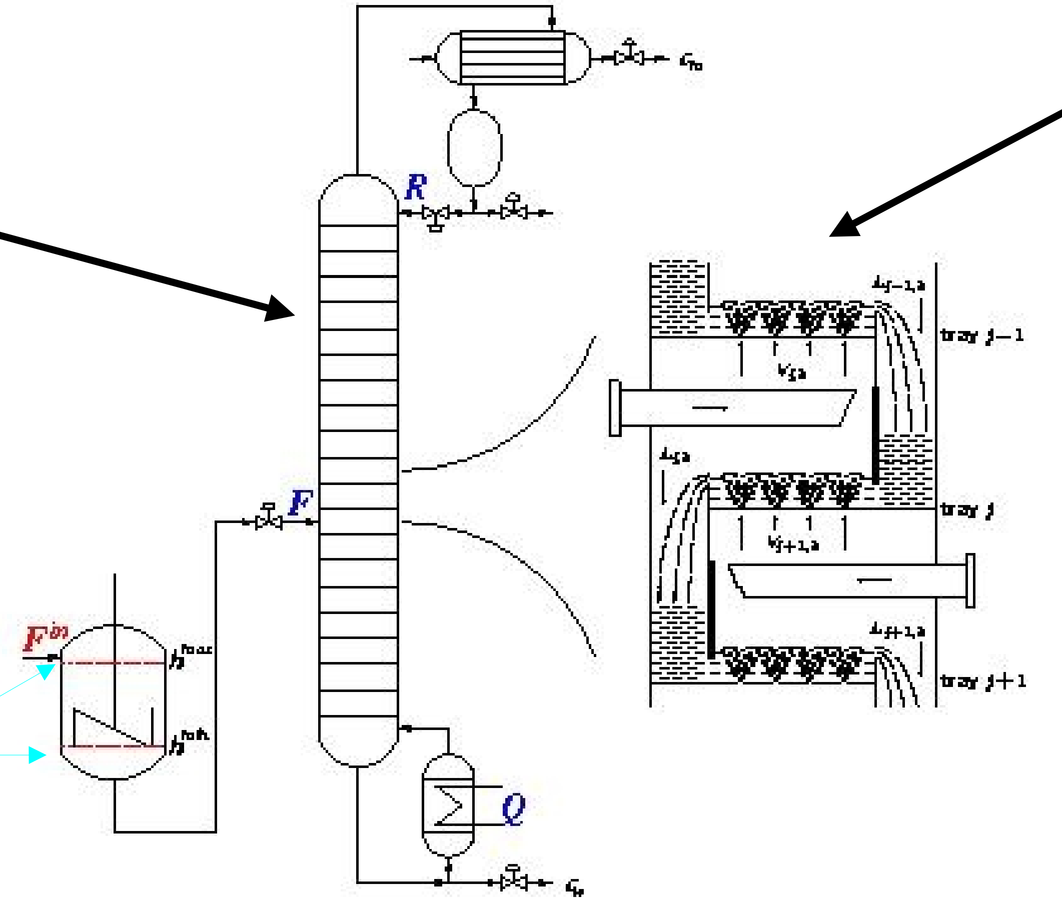
R.H., P. Li, A. Möller, M. Wendt, G. Wozny (Comp.&Math. Appl. 2003)

Distillation
(Methanole-Water)

Modeled by large
DAE-System

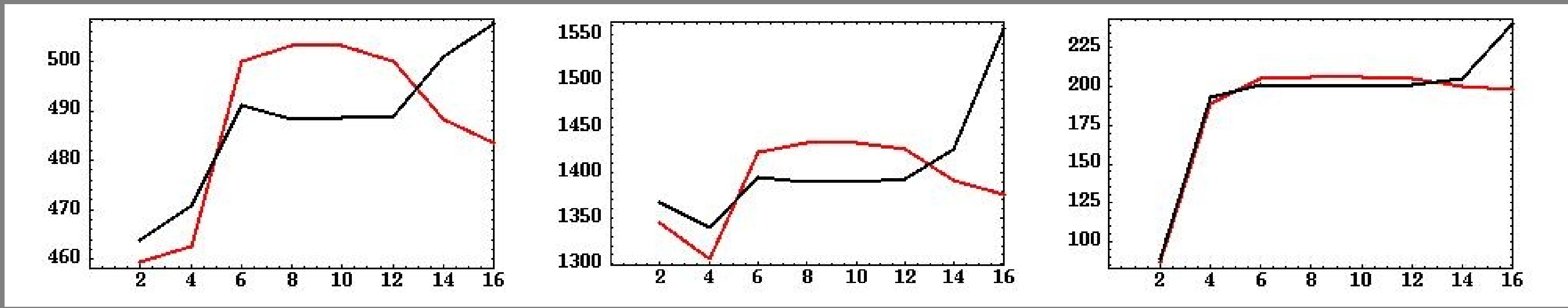
continuous
(stochastic) inflow

level constraints
(as chance constraints)



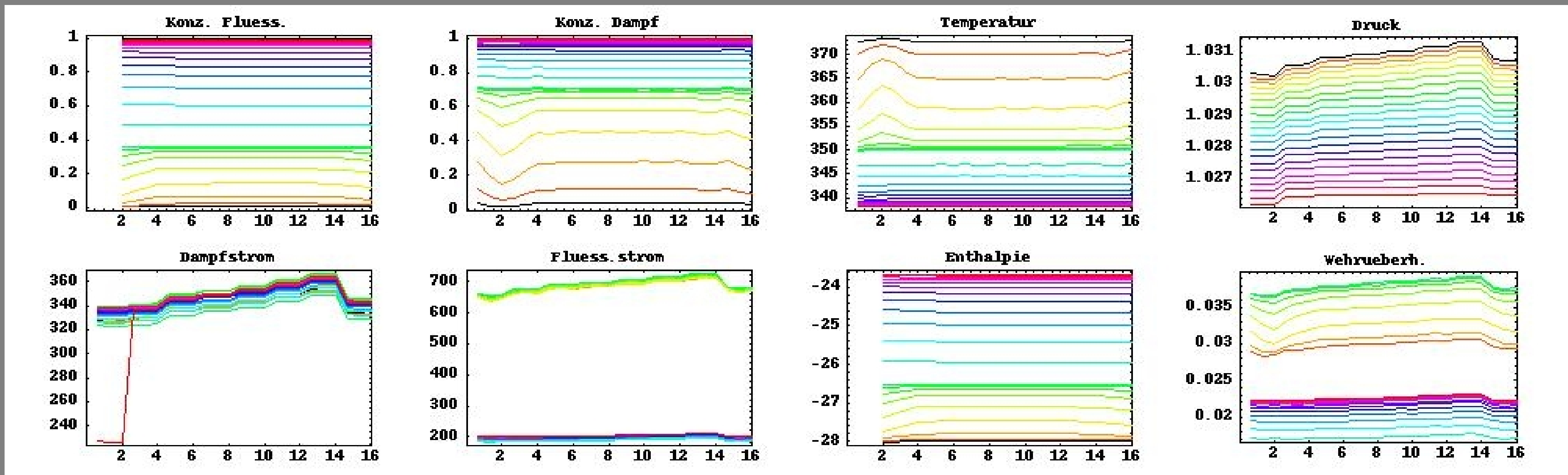
Optimal Control

feed rate F heating Q reflux R



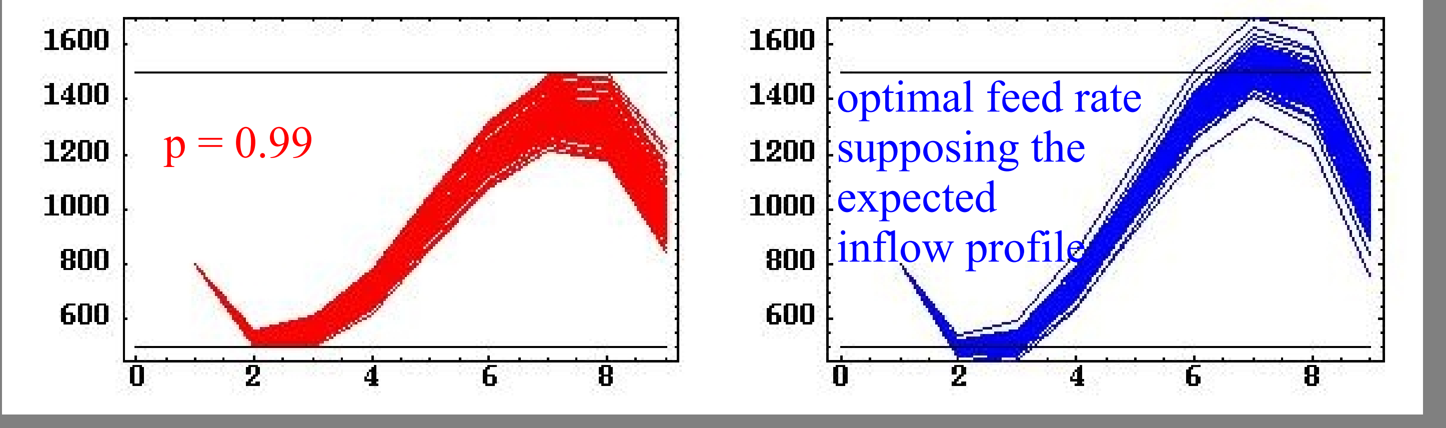
p = 0.90/0.99

8 selected state variables in 20 trays of column

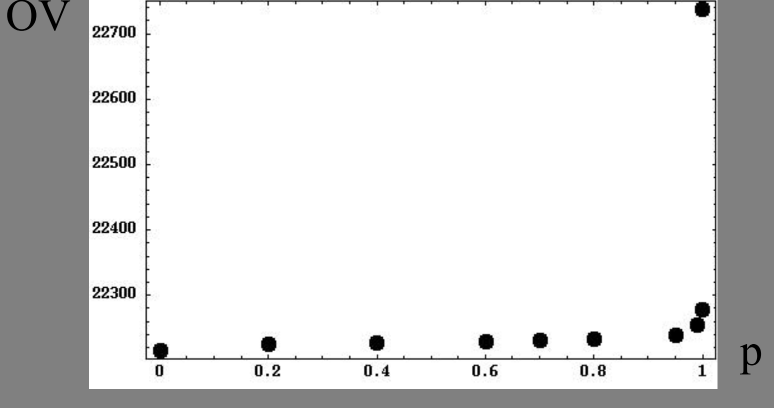


Simulated Inflow Profiles

Application of optimal feed rate to 100 simulated inflow profiles (Gaussian Process)



Optimal value (OV) of the objective versus safety level (p)



Stochastic Reservoir Constraints

$\xi = (\xi_1,\dots,\xi_n)$ discretized stochastic inflow to reservoir

$x = (x_1,\dots,x_n)$ discretized control of feed extraction from reservoir

l_0 initial filling level of reservoir

F upper filling level allowed for reservoir

Stochastic reservoir constraints:

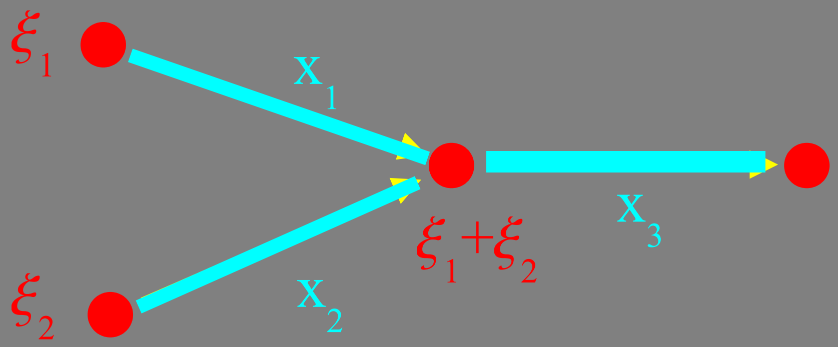
$$\underbrace{l_0+\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i}x_j}_{\text{filling level at time } i} \leq F \quad (i=1,\dots,n)$$

Chance Constraints: $P(L\xi \leq \alpha(x)) \geq p$

regular, triangular

Capacity Expansion in Stochastic Networks

Graph with stochastic load ξ in the nodes and with capacities x in the arcs



$$\xi_1 \leq x_1$$
$$\xi_2 \leq x_2$$
$$\xi_1+\xi_2 \leq x_3$$

Chance Constraint Model: $P(A\xi \leq x) \geq p$

number of inequalities larger than dimension of random vector

→ A not surjective!

3 Basic Types of Chance Constraints (CC)

CC with stochastic coefficient matrix: $P(\Xi x \leq a) \geq p$

(e.g., portfolio optimization, mixture problems)

CC with separated randomness: $P(A\xi \leq \alpha(x)) \geq p$

a) regular case: A *surjective* (e.g., stochastic reservoir-constraints)

b) singular case: A *not surjective*

occurs when number of inequalities exceeds dimension of random vector

(e.g., networks with stochastic demand,
robotics: collision avoidance with random obstacles)

Structural Properties of CCs with Stochastic Coefficient Matrix

R.H. (SPEPS 2005)

Theorem: (Van de Panne/Popp, Kataoka, 1963)

Let $\xi \sim N(\mu, \Sigma)$, Σ *positive definite.*

Then, $M := \{x \in \mathbb{R}^n | P(\langle x, \xi \rangle \leq \alpha) \geq p\}$ *is convex for* $p \geq 0.5$.

Extended Model: $M_\alpha^p := \{x \in \mathbb{R}^n | P(\langle q(x), \xi \rangle \leq \alpha) \geq p\} \quad (\alpha \in \mathbb{R})$

An extended convexity result

$$M_\alpha^p := \{x \in \mathbb{R}^n \mid P(\langle q(x), \xi \rangle \leq \alpha) \geq p\}$$

Theorem:

$Let\ \ \xi\ \ have\ \ an\ \ elliptically-symmetric\ \ distribution\ \ with\ \ density$

$and\ \ parameters\ \ \theta, \Sigma\ \ (positive\ \ definite)$

$Under\ \ each(!)\ \ of\ \ the\ \ following\ \ assumptions:$

1. $q\ \ is\ \ affine-linear$

or

2. $q_i \geq 0,\ \ convex,\ \ \theta_i \geq 0,\ \ \Sigma_{i,j} \geq 0$

$it\ \ holds\ \ that\ \ M_\alpha^p\ \ is\ \ convex\ \ for\ \ alle\ \ \alpha \in \mathbb{R}\ \ and\ \ all\ \ p \geq 0.5.$

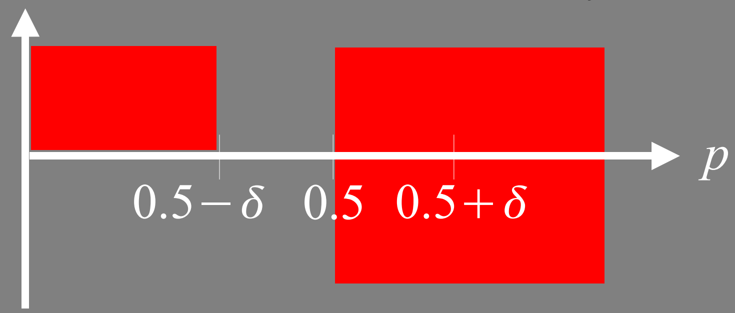
Exact domains of convexity, nontriviality and compactness

$M_\alpha^p := \{x \in \mathbb{R}^n \mid P(\langle q(x), \xi \rangle \leq \alpha) \geq p\}, \; \xi \sim N(\mu, \Sigma), \; \Sigma$ *positive definite*

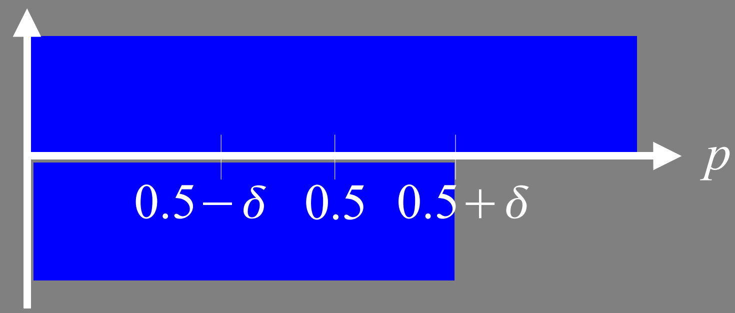
$\delta := \Phi(\|\mu\|_{\Sigma^{-1}}) - 0.5$

Theorem: *Let q be affine linear and regular. Then,*

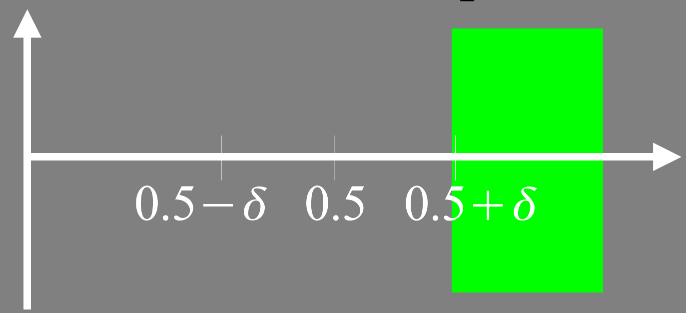
domain of convexity



domain of nontriviality



domain of compactness



Existence of Solutions to Problems with Stochastic Coefficient Matrix

Optimization problem: $(P) \quad min\{ f(x) | P(\Xi q(x) \leq a) \geq p\} \quad p \in (0,1)$

Theorem:

- $f \quad lower\ semicontinuous$
- $\xi_i \sim N(\mu_i, \Sigma_i), \ \xi_i = \ rows\ of\ \ \Xi, \ \Sigma_i \ \ positive\ \ definite$
- $a \ \geq \ 0$
- $q \ \ homeomorphism \ \ (e.g., \ \ q(x) = x)$

$\longrightarrow (P) \ \ has\ \ a\ \ solution\ \ for\ \ p \ > \ \underbrace{\min_i \ \ \Phi(\|\mu_i\|_{\Sigma_i^{-1}})}_{<\ 1}$

Singular Normal Distributions and Quasiconcave Measures

CC with separated randomness: $P(A\xi \leq \alpha(x)) \geq p$

$$P(A\xi \leq \alpha(x)) \geq p \;\Leftrightarrow\; F_\eta(\alpha(x)) \geq p \quad (with \;\; \eta := A\xi)$$

distribution function of η

Special case: normal distribution: $\xi \sim N(\mu, \Sigma) \;\Rightarrow\; \eta \sim N(A\mu, A\Sigma A^T)$

regular

regular case: $A \;\; surjective \;\Rightarrow\; A\Sigma A^T \;\; regular$

singular case: $A \;\; not \;\; surjective \;\Rightarrow\; A\Sigma A^T \;\; singular$

$$\Rightarrow F_\eta \;\; is \;\; a \;\; singular \;\; normal \;\; distribution$$

Lipschitz continuity of quasiconcave Measures

R.H., W. Römisch (Ann. Oper. Res. 2005)

Theorem:

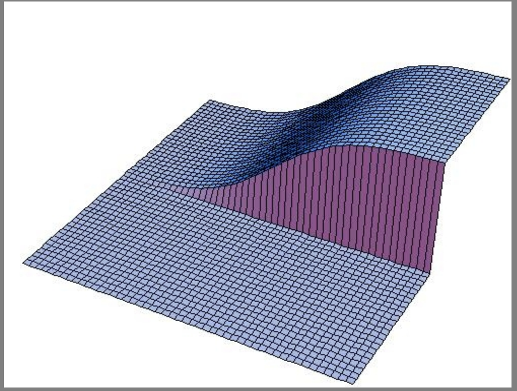
Let ξ have a *quasiconcave* distribution (e.g., regular or singular normal, uniform, Pareto, Dirchlet, lognormal, Gamma etc.).

Then, the distribution function of ξ is (Lipschitz) continuous iff

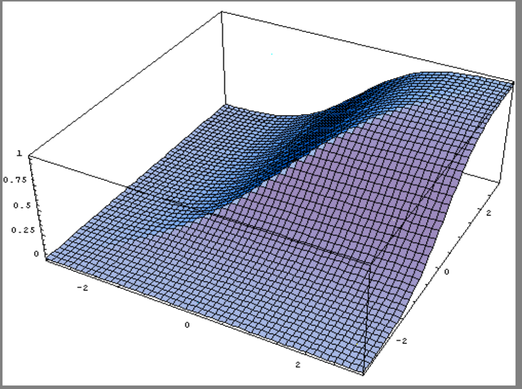
$$Var\,\xi_i \neq 0 \quad \forall i.$$

Example: Singular normal distribution with covariance matrix of rank 1

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$



Reduction of singular normal distribution functions to regular ones

J. Bukszar, R.H., M. Hujter, T. Szantai (SPEPS 2004)

$\eta \sim N(\mu, \Sigma)$, $\Sigma = AA^T$, $M(x) = \{u \mid Au \le x\}$, $I(u,x) = \{i \mid \langle a_i, u\rangle = x_i\}$

singular — Cholesky-decomposition — induced polyhedron — active index set at u

$I_{M(x)} = \{I \mid \exists u \in M(x) : I(u,x) = I\}$

family of all active index sets

x *is called* ***nondegenerate***, *if* $rank\{a_i \mid i \in I\} = |I| \quad \forall I \in I_{M(x)}$

Theorem:

If $x - \mu$ *nondegenerate* $\Rightarrow$ $F_\eta(x) = 1 + \sum_{I \in I_{M(x-\mu)}} (-1)^{|I|} F_{-\eta^I}(-x^I)$

$-\eta^I \sim N(-\mu^I, A^I (A^I)^T)$

regular

singular normal distribution ($F_\eta(x)$)

sum of regular normal distributions

Gradient Formula and Algorithm

$$If \quad x-\mu \quad nondegenerate \;\Rightarrow\; F_\eta(x) = 1 + \sum_{I \in I_{M(x-\mu)}} (-1)^{|I|} F_{-\eta^I}(-x^I)$$

$$F_{-\eta^I} \quad smooth \quad (regular\ normal\ distribution\ function)$$

$$x-\mu \quad nondegenerate \;\Rightarrow\; I_{M(x-\mu)} \quad locally \quad constant$$

Corollary:

$$If \quad x-\mu \quad nondegenerate \;\Rightarrow\; \nabla F_\eta(x) = \sum_{I \in I_{M(x-\mu)}} (-1)^{|I|+1} \nabla F_{-\eta^I}(-x^I)$$

Algorithm:

1. *Determine all corners of* $M(x-\mu)$ *(Fukuda, Scdd+)*
2. *Compute* $I_{M(x-\mu)}$ *as family of subsets of corner index sets*
3. *Calculate the* F_η *by formula from regular normal distributions* $F_{-\eta^I}$ *(Genz, Szantai)*
4. *Calculate gradients by reduction to functional values*

Numerical Tests

Dim:	**5 x 10**	Dim:	**5 x 15**	Dim:	**5 x 20**	Dim:	**5 x 25**
Prob.:	0.982894	Prob.:	0.942901	Prob.:	0.947281	Prob.:	0.954225
Error:	0.000001	Error:	0.000002	Error:	0.000002	Error:	0.000002
Poly:	1.00	**Poly:**	1.00	**Poly:**	1.00	**Poly:**	1.00
Szantai:	1599.89	Deak:	16200.45	Deak:	7594.09	Szantai:	597.93
Deak:	24579.00	Szantai:	105322.83	Szantai:	27126.86	Deak:	960.76
MC:	1715262.63	MC:	720219.79	MC:	360681.66	MC:	48868.45
Genz:	2236403.28	Genz:	1422991.07	Genz:	614578.55	Genz:	58847.63

Dim:	**10 x 20**	Dim:	**10 x 25**	Dim:	**15 x 20**	Dim:	**15 x 25**
Prob.:	0.974131	Prob.:	0.977193	Prob.:	0.940839	Prob.:	0.987551
Error:	0.000001	Error:	0.000001	Error:	0.000003	Error:	0.000003
Poly:	1.00	**Poly:**	1.00	**Poly:**	1.00	**Szantai:**	1.00
Szantai:	174.41	Szantai:	16.08	Szantai:	2.79	Deak:	55.42
Deak:	217.08	Deak:	67.87	Deak:	4.37	MC:	204.00
MC:	1888.59	MC:	601.72	MC:	16.68	Genz:	373.16
Genz:	3852.22	Genz:	1583.00	Genz:	21.22	Poly:	not available

Stabilität in Problemen mit separiertem Zufall

R.H., W. Römisch (Math. Prog. 2004)

gegebenes Optimierungsproblem: $(P) \quad min\{f(x) | x \in X, F_\eta(Ax) \geq p\}$

Verteilung von η oft unbekannt.

Approximation durch ξ z.B. auf der Basis historischer Daten.

Stabilität von Lösungen und Optimalwerten in (P) bei kleinen Störungen?

Maß für Störung (Kolmogorov-Abstand): $d_K(\eta, \xi) = sup_{z \in \mathbb{R}^n} |F_\eta(z) - F_\xi(z)|$

Lösungsmengenabbildung: $\Psi(\xi) = argmin\{f(x) | x \in X, F_\xi(Ax) \geq p\}$

Optimalwertfunktion: $\varphi(\xi) = inf\{f(x) | x \in X, F_\xi(Ax) \geq p\}$

$\xi = \eta \Rightarrow Originalproblem$

Qualitative und quantitative Stabilität

$$(P)\quad min\{f(x) | x \in X, F_\eta(Ax) \geq p\}$$

$$f \quad konvex, \quad X \quad konvex\, und\, abgeschlossen, \quad \log F_\eta \quad konvex$$

für viele multivariate Verteilungen erfüllt (Prekopa)

Theorem:

- Lösungsmenge von (P) nichtleer und beschränkt
- $\exists \bar{x} \in X: \quad F_\eta(A\bar{x}) > p \quad (Slaterpunkt)$

→

1. $\Psi \quad ist \quad oberhalbstetig \quad in \quad \eta$
2. $|\varphi(\xi) - \varphi(\eta)| \leq L\, d_K(\xi, \eta) \quad (lokal)$

- Falls zusätzlich:

$$f \quad linear - quadratisch, \quad X \quad Polyeder, \quad \log F_\eta \quad stark \quad konvex$$

→

3. $d_H(\Psi(\xi), \Psi(\eta)) \leq L[d_K(\xi, \eta)]^{1/2} \quad (lokal)$

Exponentielle Schranken für empirische Approximationen

Ausgangsproblem: $min\{f(x)|x\in X, F_{\eta}(Ax)\geq p\}$

Sei $\{\xi_N\}\sim\mu$ eine Folge unabhängiger Zufallsvektoren.

Empirische Approximation: $\mu_N := N^{-1}\sum_{i=1}^{N}\delta_{\xi_i}$

Dvoretzky-Kiefer-Wolfowitz Ungleichung: $P(d_K(\mu_N,\mu)>\delta)\leq C_1\exp(-C_2\delta^2 N)$

Unter den Voraussetzungen des qualitativen Stabilitätsresultates folgt:

(1) $\forall\varepsilon>0\ \forall x_N\in\Psi(\mu_N):\ P(d(x_{N,}\Psi(\mu))\leq\varepsilon)\ \underset{N}{\rightarrow}\ 1.$

(2) $\forall\varepsilon\in(0,\varepsilon_0):\ P(|(\varphi(\mu_N)-\varphi(\mu))|>\varepsilon)\ \leq\ 2C_1\exp(-C_2 N\varepsilon^2 L^{-2})$

Unter den Voraussetzungen des quantitativen Stabilitätsresultates folgt:

(3) $\forall\varepsilon\in(0,\varepsilon_0):\ P(d_H(\Psi(\mu_N),\Psi(\mu))>\varepsilon)\ \leq\ 2C_1\exp(-C_2 N\varepsilon^4 L^{-4})$